THE ORIGIN AND EVOLUTION OF COSMOLOGICAL STRUCTURE

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Below are the lecture notes corresponding to the Cosmological Perturbation Theory, Inflation, and Evolution of Structures chapters of the Cosmology Course in the MSc in Particle Physics and Cosmology, offered by the University of Cantabria and the International University Menendez Pelayo. These notes are heavily based on chapters 3, 5-9 from *Modern Cosmology* (Second Edition), by Dodelson and Schmidt [3], and the seminal paper on cosmological perturbation theory by Ma and Bertschinger, 1995 [4]. These notes are under construction, and are expected to significantly grow (i.e., covering the whole course of cosmology) and evolve in the future. Throughout, natural units $c = \hbar = 1$ will be used unless otherwise stated.

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CHAPTER 1

COSMOLOGICAL PERTURBATION THEORY

Cosmology deals with the nature and evolution of the Universe and its components in a statistical manner, therefore it is at its core the application of general relativity and statistical mechanics, combined with the astrophysics that drive the physics of the tracers that we can observe. However, we will ignore the latter for the time being and focus on the statistical properties of matter and radiation in the Universe, and how they affect and are affected by gravity,[1] which is the only relevant long-range force that we will consider.

Some approaches prefer to treat some of the components of the Universe as fluids. Instead, we will treat each component from a statistical point of view: since we do not care about the behavior of individual particles, but their collective properties, all the information that we need is enclosed in the distribution function f of the number of particles N in an infinitesimal

[1]Studies of general relativity usually refer as *matter* to all components that appear in the right-hand side of the Einstein equations, i.e., including also radiation. We may use that language in certain situations.

phase-space element around position $\boldsymbol{x}$ and momentum $\boldsymbol{p}$, such as

$$N(\boldsymbol{x},\boldsymbol{p},t) = f(\boldsymbol{x},\boldsymbol{p},t)\mathrm{d}^3\boldsymbol{x}\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,, \tag{1.1}$$

where we assume that the number of particles is large enough for f to approach the continuous limit. The $(2\pi)^3$ factor appears because by Heisenberg's principle, no particle can be localized into a region of phase space smaller than $(2\pi\hbar)^3$, which makes it the size of the fundamental element. The equations describing the evolution of f as function of time and phase-space coordinates are the Boltzmann equations.

As we will see, the Boltzmann Equations already include the continuity and Euler equation that are usually applied to describe the dynamics of fluids for cosmological perturbation theory, but in addition provide a framework to straightforwardly include any additional interaction between the particles of the fluid or between different components of matter and radiation. Furthermore, some components impact cosmological perturbations beyond their density, velocity and anisotropic stress (the monopole, dipole and quadrupole of the phase-space distribution), and higher-order moments, not considered in the continuity and Euler equations, must be taken into account. This is why we prefer to develop the cosmological perturbation theory with full generality, and then specify the properties for each component.

1.1 Boltzmann Equations

A system of particles is statistically determined by its distribution function f in phase space, so we just need an equation that describe its evolution. Neglecting for now any interaction between particles (e.g., scatter, decays, annihilation, etc), the total number of particles must be conserved. This case is referred to as 'collisionless' in the context of the Boltzmann equations. Therefore, the *total* (rather than *partial*) time derivative of the distribution function must vanish:

$$\frac{\mathrm{d}f(\boldsymbol{x},\boldsymbol{p},t)}{\mathrm{d}t} = 0\,; \qquad \text{where}\ \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{\boldsymbol{x}}\cdot\boldsymbol{\nabla}_x + \dot{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_p\,, \tag{1.2}$$

where dot denote time derivatives and the subscript of the gradients denote the arguments they must be taken with respect to. The forces driving the problem at hand are included substituting the equations of motion in the expression above. But before that, we need to generalize this expression to the case of an expanding Universe.

One of the main benefits of working in terms of the distribution function f is that we can use it to derive all macroscopic properties of the particles under study. In all generality, the relativistic energy-momentum tensor is

$$T^\mu_\nu(\boldsymbol{x},t) = \frac{g_*}{\sqrt{-\det(g_{\alpha\beta})}}\int\frac{\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3}{(2\pi^3)}\frac{P^\mu P_\nu}{P^0}f(\boldsymbol{x},\boldsymbol{p},t)\,, \tag{1.3}$$

where g_* accounts for all the degenerate particle state that are described by f (e.g., $g_* = 2$ for a particle with spin 1/2) and P^μ is the four-momentum, defined in terms of the affine parameter of the geodesic with λ (to avoid confusion with the shear stress σ, which will be introduced at the end of this chapter) as

$$P^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \,. \tag{1.4}$$

Equation (1.3) shows that the energy-momentum tensor gives the current density of the four-momentum carried by the particles with distribution function f. The momentum integral over f gives you the number density; weighted by P_ν it gives you the four-momentum density; and additionally weighted by the four-velocity P^μ/P^0 gives you the current density of the four-momentum. Finally, the prefactor is a geometric factor to ensure the conservation of the energy-momentum tensor: $\nabla_\mu T^\mu_\nu = 0$.

We will first consider a smooth Universe, expanding according to the FLRW metric. However, f still depends on a six-dimensional phase space: we will track time separately as before, and we can express P^0 as function of $\boldsymbol{p}$ using the norm p of the three-momentum and the mass-shell constraint. Then, for the FLRW metric, we have

$$\left(P^0\right)^2 \equiv E^2 = p^2 + m^2 \,. \tag{1.5}$$

Furthermore, it is convenient to separate the dependence on $\boldsymbol{p}$ into a dependence on its magnitude p and the unitary vector $\hat{p}^i = \hat{p}_i$ which determines its direction and satisfies $\delta_{ij}\hat{p}^i\hat{p}^j = 1$.[2] Since $\hat{p}^i$ is expected to be proportional to P^i, such as $P^i = \mathcal{C}\hat{p}^i$; then

$$p^2 = g_{ij}P^iP^j = g_{ij}\hat{p}^i\hat{p}^j\mathcal{C}^2 = a^2\delta_{ij}\hat{p}^i\hat{p}^j\mathcal{C}^2 \Rightarrow \mathcal{C} = \frac{p}{a} \Rightarrow P^i = \frac{p}{a}\hat{p}^i \,, \tag{1.6}$$

and we can interchange always P^i by p and $\hat{p}^i$. Therefore, we can generalize $f(\boldsymbol{x}, \boldsymbol{p}, t) = f(x^i, \hat{p}^i, p, t)$ and express the Boltzmann Equation as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} = 0 \,. \tag{1.7}$$

This is the most general expression of the Boltzmann equation in the absence of interactions between particles. In the rest of the section we will discuss a specific simple case and discuss generally the source term that encloses the microphysics determining the particle interactions.

1.1.1 Boltzmann Equation in FLRW

Let us specify the Boltzmann Equation for a smooth expanding Universe, as the one described by the FLRW. In this scenario, the direction of the

[2] In general, we will use hats as the notation to denote unitary 3-vectors.

momentum does not change, hence we can drop the last term in Eq. (1.7). The term than depends on $\partial f/\partial x^i$ could also be dropped (the background is homogenous and isotropic), but it is easy to handle and will be useful once we add perturbations. We need to obtain then the values of the total derivatives of x^i, and p with respect to time. Using Eq. (1.4), we get

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} = P^i \frac{1}{P^0} = \frac{p}{E}\frac{\hat{p}^i}{a}\,. \tag{1.8}$$

In order to obtain the total derivative of p with respect to time we start from the time component of the geodesic,

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta = -a^2 H \delta_{ij} P^i P^j\,, \tag{1.9}$$

where the last equality relies on the Christoffel symbols for the FLRW metric. Since $P^0 = \mathrm{d}t/\mathrm{d}\lambda$, we have (multiplying and deriving by $\mathrm{d}t$)

$$P^0 \frac{\mathrm{d}P^0}{\mathrm{d}t} = p\frac{\mathrm{d}p}{\mathrm{d}t} = -Hp^2 \rightarrow \frac{\mathrm{d}p}{\mathrm{d}t} = -Hp\,, \tag{1.10}$$

where the first equality is obtained from applying the time derivative to Eq. (1.5) in the form $\mathrm{d}(P^0)^2/\mathrm{d}t = 2P^0\mathrm{d}(P^0)/\mathrm{d}t = \mathrm{d}(E^0)^2/\mathrm{d}t = 2p\mathrm{d}p/\mathrm{d}t$. The equation above shows that the physical momentum of any particle decays as $1/a$ in a smooth expanding Universe. Then the collisionless Boltzmann equation in an unperturbed expanding Universe is given by

$$\frac{\partial f}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - Hp\frac{\partial f}{\partial p} = 0\,. \tag{1.11}$$

Similarly, we can derive the evolution of the number density, since it is the momentum integral of f, as discuss above. For a homogeneous Universe (i.e., $\partial f/\partial x^i = 0$), and integrating by parts the momentum component as

$$H\int \frac{\mathrm{d}^2\Omega_\mathrm{p}}{(2\pi)^3}\int_0^\infty p^2\mathrm{d}p p\frac{\partial f}{\partial p} = -3H\int \frac{\mathrm{d}^2\Omega_\mathrm{p}}{(2\pi)^3}\int_0^\infty p^2\mathrm{d}p f = -3Hn\,, \tag{1.12}$$

where Ω_p is the solid angle for the momentum vector and we have used that for any regular distribution function fp^3 vanishes at $p = 0$ and $p = \infty$, we find

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = 0\,. \tag{1.13}$$

This is, in the absence of collisions, the number density decays as a^{-3} in a homogeneous expanding Universe. However, collisions can change this behaviour, as well as the evolution of the distribution function. We briefly introduce below the collision term.

1.1.2 Collision terms

So far we have studied the evolution of the distribution function for particles that do not interact between them or have any interaction with other components beyond long-range forces (e.g., gravity). However, when these conditions do not apply there is a source term in the Boltzmann equation, called the collision term.[3] As it is expected, the collision term depends on the actual phase-space distribution of the particles involved; hence, in general, the Boltzmann equation becomes

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C[f]\,. \tag{1.14}$$

In order to show in a simple example how to derive the collision term, let us consider a reaction where particles of type (1) and (2) interact to form particles of type (3) and (4):

$$(1)_{\boldsymbol{p}} + (2)_{\boldsymbol{q}} \longleftrightarrow (3)_{\boldsymbol{p}'} + (4)_{\boldsymbol{q}'}\,, \tag{1.15}$$

where the subscripts denote each particle's momenta.[4] Of course, the reaction conserves energy and momentum, and each particle has its own distribution function f_s, with some states that can be degenerate (e.g., often in cosmology, the spin does not play an active role, hence instead of tracking it directly, we weight the distribution function with a suitable degeneracy weight g_*).

We assume that this reaction is local, e.g., the reaction occurs at $(\boldsymbol{x}, t)$ and we only need to determine the momenta. Furthermore, we need to compute the collision term for each independent particle type (which most likely will couple the evolution equations for the fours types of particles).

At the end of the day, the collision term (say, for particles of type 1), as a source term, accounts for all particles that get scattered away from $\boldsymbol{p}$ by the forward reaction (subtract them from $f_1(\boldsymbol{x}, \boldsymbol{p}, t)$) and all particles that get scattered to $\boldsymbol{p}$ by the backward reaction (add them to $f_1(\boldsymbol{x}, \boldsymbol{p}, t)$). The forward and backward reaction rates are determined by the scattering amplitude $|\mathcal{M}|^2$, which can be computed using Feynman diagrams, and the number of particles of each type with the momenta required. In this case we have the products $f_1(\boldsymbol{p})f_2(\boldsymbol{q})$ and $f_3(\boldsymbol{p}')f_4(\boldsymbol{q}')$ for the forward and backward reactions. We need to account for stimulated emission (i.e., Bose enhancement) and Pauli exclusion principle (i.e., Pauli blocking), too, which amounts to include factors of $(1 \pm f_3(\boldsymbol{p}))(1 \pm f_4(\boldsymbol{q}'))$ to the forward reaction (and equivalently to the backward reaction), depending on whether the particle involved is a

[3] In the context of the Boltzmann equation, the effect of direct particle interactions is referred to as 'collisions', and it is a way to describe the microphysics driving the particle interaction in an effective statistical way.

[4] This reaction can be of scattering and annihilation, depending on the nature of particles 3 and 4 with respect to particles 1 and 2. The derivation of this subsection can straightforwardly be extended to other cases involving a different number of particles.

fermion or a boson. What matters is the occupation of the phase-space element in the *result* state from each reaction; this is why they are interchanged. If the particle is a boson the reaction is enhanced, since bosons occupying the same state are favored, while if the particle is a fermion, if a specific state is occupied the reaction cannot happen. Finally, the conservation of momentum and energy is enforced using corresponding Dirac delta functions.

In order to consider the whole phase space in position $(\boldsymbol{x}, t)$ that affects particle 1 with momentum $\boldsymbol{p}$ we need to integrate over all momenta of particles 2, 3, and 4. However, there is a small subtlety: in a relativistic setting, phase-space integrals are four-dimensional (three momentum components and the energy), but energy and momentum are related by the mass-shell constraint, therefore

$$\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \int \mathrm{d}E\,\delta_{\mathrm{D}}^{(1)}(E^2 - p^2 - m^2) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \int \mathrm{d}E \frac{\delta_{\mathrm{D}}^{(1)}\left(E - \sqrt{p^2 + m^2}\right)}{2E}, \tag{1.16}$$

which adds a factor of $1/2E$ after integrating over the energy.

Taking all these considerations into account, the collision term becomes

$$\begin{aligned} C[f_1(\boldsymbol{p})] = \frac{1}{2E_1(p)} \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3 2E_2(q)} \int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3 2E_3(p')} \int \frac{\mathrm{d}^3\boldsymbol{q}'}{(2\pi)^3 2E_4(q')} |\mathcal{M}|^2 \times \\ \times (2\pi)^4 \delta_{\mathrm{D}}^{(3)}(\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{p}' - \boldsymbol{q}')\delta_{\mathrm{D}}^{(1)} \left[E_1(p) + E_2(q) - E_3(p') - E_4(q')\right] \times \\ \times \{f_3(\boldsymbol{p}')f_4(\boldsymbol{q}') \left[1 \pm f_1(\boldsymbol{p})\right] \left[1 \pm f_2(\boldsymbol{q})\right] - \\ - f_1(\boldsymbol{p})f_2(\boldsymbol{q}) \left[1 \pm f_3(\boldsymbol{p}')\right] \left[1 \pm f_4(\boldsymbol{q}')\right]\} . \end{aligned} \tag{1.17}$$

1.2 Perturbed Universe

So far, we have considered only a smooth expanding Universe described by the FLRW metric. This is enough to study the background expansion and thermal history of the Universe, but the Universe has small inhomogeneities that grow over time and host the galaxies and large scale structure that we observe today. Fortunately to us, these inhomogeneities are very small, which allows us to treat them perturbatively. In particular, the linear approximation will be enough except for the small scales at late times.

As we have discussed previously in the context of a smooth Universe, the metric perturbations and the perturbations in the phase-space distributions of the matter components are coupled. Metric perturbations produce perturbations in the background properties of matter and radiation, which in turn affect the metric perturbations, as expected from the energy-momentum tensor term in the Einstein equations and the presence of gravity (through the time derivatives of position and momentum) in the Boltzmann equations. Therefore, we need to treat perturbatively both set of equations to derive the system that describes the evolution of cosmological perturbations.

In what follows we will assume a flat Universe described by the ΛCDM model and where general relativity is an accurate description of gravity, with dark matter, baryons, neutrinos and photons. Also, for convenience, we will work with the conformal time τ, which is related with the physical time through $\mathrm{d}t = a\mathrm{d}\tau$. Derivatives with respect to conformal time are denoted with a prime, and $a'/a = \mathcal{H}$ is the conformal Hubble parameter. Using the conformal time, the FLRW metric satisfies

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] . \tag{1.18}$$

Consider now a small perturbation to this metric, which can be written in full generality as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where the bar denotes background values. Since all these perturbations are very small, we can restrict our work to linear order in perturbation theory. Therefore, we can consider the perturbations a three-tensors and raise and lower indices in the spatial indices always with Kronecker delta (this is not the case for four-vector indices).

The perturbation (actually, any tensor) can be decomposed in scalar, vector and tensor contributions. The decomposition theorem is a very important result in general relativity (which we will not prove here, can be found in Ref. [3]), which states that perturbations of each type evolve independently at linear order. Taking this into account, let us express the perturbed metric as

$$\begin{aligned} g_{00} &= -a^2(\tau)\left\{1 + 2\Psi(\boldsymbol{x},\tau)\right\} , \qquad\qquad g_{0i} = a^2(\tau)w_i(\boldsymbol{x},\tau)\,, \\ g_{ij} &= a^2(\tau)\left\{\left[1 + 2\Phi(\boldsymbol{x},\tau)\right]\delta_{ij} + \chi_{ij}(\boldsymbol{x},\tau)\right\} , \end{aligned} \tag{1.19}$$

where Ψ and Φ are scalars, w_i is a vector and χ_{ij} is a symmetric trace-free tensor ($\delta^{ij}\chi_{ij} = 0$); χ_{ij} can be taken to be traceless since any trace can be reabsorbed into Φ.[5]

We can further decompose w_i and χ_{ij}. Vectors can be decomposed in longitudinal (curl-free) and transverse (divergence-free) components, such as

$$w_i = w_i^{\parallel} + w_i^{\perp} ; \qquad w_i^{\parallel} \equiv \partial_i w ; \qquad \partial_i w_i^{\perp} = \epsilon^{ijk}\partial_j w_k = 0\,, \tag{1.20}$$

where actually w is a scalar variable. Therefore, a vector can be decomposed into a scalar function and a transverse component.

Similarly, $\chi_{ij} = \chi_{ij}^{\parallel} + \chi_{ij}^{\perp} + \chi_{ij}^{\mathrm{T}}$, which are respectively the longitudinal, solenoidal and traceless-transverse parts. The first two components (the divergences of which are longitudinal and transverse vectors) can be written in terms of a scalar χ and a transverse vector χ_i, such as

$$\chi_{ij}^{\parallel} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\chi\,, \qquad\qquad \chi_{ij}^{\perp} = \partial_i\chi_j + \partial_j\chi_i\,, \tag{1.21}$$

[5]A warning regarding different references is in place regarding this decomposition and the derivation below. Different references use different conventions (signature of the metric, signs of the perturbations, notation of the perturbations, etc), which may impact final expressions in factors and signs for each of the perturbations.

and these three components satisfy

$$\epsilon^{ijk}\partial_j\partial^l\chi^{\parallel}_{lk} = \partial^i\partial^j\chi^{\perp}_{ij} = \partial^i\chi^{\mathrm{T}}_{ij} = 0\,. \tag{1.22}$$

Note that this decomposition is not unique. We have decomposed the most general metric perturbation into four scalar components (Φ, Ψ, w and χ), each having a degree of freedom, two transverse-vector components ($w_i^{\perp}$ and χ_i), each with two degrees of freedom, and symmetric trace-free divergence-free tensor (χ^{T}_{ij}), which two degrees of freedom, for a total of 10 degrees of freedom.

1.2.1 Fourier-space computations

Before going deeper in the choice of coordinates and the gauge problem in perturbation theory, let us step back to discuss the benefits of working in Fourier space (rather than in configuration space), determined by wavevectors $\boldsymbol{k}$. As reference, we follow the Fourier convention

$$\tilde{f}(\boldsymbol{k}) = \int \mathrm{d}^3\boldsymbol{x}\; f(\boldsymbol{x})e^{-i\boldsymbol{kx}}, \qquad f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\; \tilde{f}(\boldsymbol{k})e^{i\boldsymbol{kx}}\,, \tag{1.23}$$

where the tilde denotes Fourier-space functions.[6] Spatial derivatives simplify significantly in Fourier space:

$$\partial_i F(\boldsymbol{x},t) = ik_i\tilde{F}(\boldsymbol{k},t)\,, \tag{1.24}$$

where $k_i = k^i$ is a 3-vector in Euclidean space. We will drop the tilde later for convenience, but the arguments and the presence of k avoids any confusion between configuration-space and Fourier-space quantities.

As an example of how working in Fourier space simplifies computations, let us consider a linear partial differential equation

$$\frac{\partial^2}{\partial t^2}\delta + A(t)\frac{\partial}{\partial t}\delta + B(t)\nabla^2 C = 0\,, \tag{1.25}$$

which in Fourier space becomes

$$\frac{\partial^2}{\partial t^2}\delta + A(t)\frac{\partial}{\partial t}\delta - B(t)k^2 C = 0\,, \tag{1.26}$$

a set of decoupled ordinary differential equations: we can solve the equation independently for each $\boldsymbol{k}$ mode, which means that every Fourier model evolves independently of the rest (instead of solving an infinite number of coupled equations in configuration space). At linear order in cosmology, each mode evolves independently of the rest, hence cosmological perturbation theory is solved in Fourier space; non linearities couple different Fourier modes, which significantly complicates the computations.

[6] As above, Fourier conventions usually lead to confusion and missing factors. Usually conventions differ in how the $(2\pi)^3$ factor is distributed between the expressions above.

1.2.2 Gauge problem

Note that the metric perturbations from Eq. (1.19) are not uniquely defined (metric perturbations must have 6 degrees of freedom and we counted 10 above) and depend on the coordinate choice. In general relativity, the choice of coordinates and fixing of degrees of freedom is called *gauge choice*.[7] Any time a metric is written, a time slicing is chosen and specific spatial coordinates are defined within it. A suitable gauge eases significantly the computations but a poor choice may complicate them and even introduce spurious perturbations can arise. This is why gauge-invariant quantities are so important: actually any cosmological observable must be gauge invariant, since physics cannot depend on the choice of coordinates.

Consider a general coordinate transformation from a coordinate system x^μ to another $\hat{x}^\mu$. Since we want to deal with small perturbations, we can Taylor expand this change of coordinates and keep only the linear displacement, so that

$$x^\mu \to \hat{x}^\mu = x^\mu + d^\mu(x^\nu)\,, \tag{1.27}$$

where the 4-vector d can be decomposed on a scalar time component α and spatial component that can in turn be decomposed into a longitudinal component $\partial^i\beta$ and a transverse component γ^i, such as

$$\hat{x}^0 = x^0 + \alpha(\boldsymbol{x},\tau)\,, \qquad \hat{x}^i = x^i + \partial^i\beta(\boldsymbol{x},\tau) + \gamma^i(\boldsymbol{x},\tau)\,. \tag{1.28}$$

This coordinate transformation results in an equivalence of the metrics:

$$g_{\mu\nu}(x) = \frac{\partial\hat{x}^\alpha}{\partial x^\mu}\frac{\partial\hat{x}^\beta}{\partial x^\nu}\hat{g}_{\alpha\beta}(\hat{x})\,. \tag{1.29}$$

The explicit metric transformation is done then by evaluating each term separately. For example, for the time-time component,

$$g_{00} = -a^2(1+2\Psi) = \frac{\partial\hat{x}^\alpha}{\partial\tau}\frac{\partial\hat{x}^\beta}{\partial\tau}\hat{g}_{\alpha\beta}(\hat{x})\,, \tag{1.30}$$

where the only term that contributes to the right-hand side is the $\alpha = \beta = 0$. This is because for other terms the derivatives are each one first-order perturbations, and hence their product is second order in perturbations (except for the time-space components, but the metric in that case is already first-order perturbation, with similar conclusion). Besides, we do not need to distinguish whether a perturbation is evaluated in one frame or another, since the difference between the coordinate systems is already a first-order perturbation, so any effect due to the point and coordinate system will be already second order and therefore negligible. Therefore, in this case we have:

$$-a^2(\tau)(1{+}2\Psi) = -(1{+}\alpha')^2a^2(\tau{+}\alpha)(1{+}2\hat{\Psi}) = -(1{+}2\alpha')(a^2(\tau){+}a'a\alpha)(1{+}2\hat{\Psi})\,, \tag{1.31}$$

[7] For this subsection we also take material from *Cosmological Dynamics*, course notes by Bertschinger [2] and the lecture notes *Cosmology III* by Baumann [1].

where the last equality only retains linear-order terms. Then, we have that $\hat{\Psi} = \Psi - \alpha' - \mathcal{H}\alpha$. Repeating this approach for all entries we find that the scalar perturbations in the two coordinate systems are related to first order in the perturbed quantities by

$$\begin{aligned} \hat{\Psi} &= \Psi - \alpha' - \mathcal{H}\alpha\,, \\ \hat{\Phi} &= \Phi - \frac{1}{3}\nabla^2\beta - \mathcal{H}\alpha \\ \hat{w} &= w + \alpha - \beta'\,, \\ \hat{\chi} &= \chi - 2\beta\,, \end{aligned} \tag{1.32}$$

while vector perturbations are related by

$$\begin{aligned} \hat{w}_i &= w_i - \gamma_i'\,, \\ \hat{\chi}_i &= \chi_i - \gamma_i\,, \end{aligned} \tag{1.33}$$

and finally the tensor perturbation is related by

$$\hat{\chi}_{ij} = \chi_{ij}\,, \tag{1.34}$$

and all quantities are evaluated at $(\boldsymbol{x}, \tau)$. For reasons that will be evident below, let us also show the transformation for $\chi_{ij}^{\parallel}$:

$$\hat{\chi}_{ij}^{\parallel} = \chi_{ij}^{\parallel} - \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\beta\,. \tag{1.35}$$

Taking into account these results, it is possible to define gauge-invariant metric perturbations, so-called Bardeen variables, that are given by:

$$\begin{aligned} \Psi^{\mathrm{B}} &= \Psi + \mathcal{H}(w - \frac{\chi'}{2}) + (w - \frac{\chi'}{2})'\,, \\ \Phi^{\mathrm{B}} &= -\Phi - \mathcal{H}(w - \frac{\chi'}{2}) + \frac{1}{6}\nabla^2\chi\,, \\ \Phi_i^{\mathrm{B}} &= \chi_i' - w_i \\ \chi_{ij}^{\mathrm{B}} &= \chi_{ij}\,. \end{aligned} \tag{1.36}$$

By choosing a coordinate frame, we fix 4 of the degrees of freedom in the metric, leaving 6 degrees of freedom. Then, what matters is what gauge optimizes the calculations for each problem. Ideally, a fully covariant treatment (i.e., using gauge-invariant quantities) is required to avoid spurious perturbations and artifacts. However, correct results are obtained when gauge-dependent quantities are converted to observables at the end of the computation, since observables must be gauge-independent by definition.

1.2.2.1 Synchronous and Conformal Newtonian gauges

Two of the most used gauges for cosmological perturbation theory are the synchronous and the conformal Newtonian gauges (also called in some contexts longitudinal gauge).

The synchronous gauge fixes $w_i = \Psi = 0$ (using the perturbations defined in Eq. (1.19)). The motivation is that in these coordinates there is a set of comoving observers who are in free fall without changing their spatial coordinates (i.e., $u^i = \mathrm{d}x^i/\mathrm{d}\lambda = 0$ is a geodesic): the so-called *fundamental* comoving observers. In this gauge, the proper or conformal time measured by a clock carried by a comoving observer and their fixed spatial coordinate *define* the coordinate system. Then there is an additional freedom to fix: the initial setting in the initial coordinate and clock for the observers. As we will see below, this residual freedom is manifested in spurious modes in the solutions for the evolution of perturbations (at super horizon modes).

The metric in synchronous gauge is then given by

$$\mathrm{d}s^2 = a^2(\tau)\left\{-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right\}\,, \tag{1.37}$$

where the metric perturbation Φ has been absorbed within h_{ij}. Limiting ourselves to scalar perturbations within this gauge, we are left with χ and $h \equiv h_{ii}$ (the trace of h_{ij}, which is proportional to Φ). In order to homogenize the notation with the literature (mostly with Ref. [4]), we introduce the field η, which is proportional to χ, such as we can write the scalar mode of h_{ij} in Fourier space as

$$h_{ij}(\boldsymbol{x},\tau) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \left\{\hat{\boldsymbol{k}}_i\hat{\boldsymbol{k}}_j h(\boldsymbol{k},\tau) + \left(\hat{\boldsymbol{k}}_i\hat{\boldsymbol{k}}_j - \frac{1}{3}\delta_{ij}\right) 6\eta(\boldsymbol{k},\tau)\right\}\,, \tag{1.38}$$

where $\boldsymbol{k} = k\hat{\boldsymbol{k}}$. Note also that $h_{ij} = h\delta_{ij}/3 + h^{\parallel}_{ij}$. Therefore, the scalar metric perturbations of interest within the synchronous gauge are h and η (note that a factor of k^2 has been absorbed in η).

The conformal Newtonian gauge (we will refer to this gauge as simply Newtonian from now on) is a particularly simple gauge, with the drawback that is limited to only scalar perturbations. In this gauge the metric is given by

$$\mathrm{d}s^2 = a^2(\tau)\left\{-(1+2\Psi)\mathrm{d}\tau^2 + (1+2\Phi)\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right\}\,, \tag{1.39}$$

which leaves the metric tensor diagonal. In this case, the metric perturbations are Ψ and Φ.

As advocated above, it is very important to know how to transform between gauges. From the definitions of these gauges and Eq. (1.32) we can relate Φ and Ψ to h and η. Let $\hat{x}^\mu$ denote the synchronous coordinates and x^μ the Newtonian coordinates with the same transformations discussed above. From the null perturbations in the synchronous gauge and Eq. (1.33) we find

$$\alpha = \beta' + \xi\,, \qquad h^{\parallel}_{ij} = -2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\beta\,, \tag{1.40}$$

where $\xi(\tau)$ is an arbitrary function of time that reflects the residual freedom mentioned above, and that we can set $\xi = 0$ without any physical impact. Thus, from Eq. (1.32) we have

$$\Psi = \beta'' + \mathcal{H}\beta' \,, \qquad \Phi = \frac{1}{6}h + \frac{1}{3}\nabla^2\beta + \mathcal{H}\beta' \,. \tag{1.41}$$

Comparing the β to $h^{\parallel}_{ij} = h_{ij} - h\delta_{ij}/3$ from Eq. (1.40) and Eq. (1.38) we find that understanding derivatives in Fourier we have

$$\beta(\boldsymbol{x}, \tau) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \frac{1}{2k^2} \left\{ h(\boldsymbol{k}, \tau) + 6\eta(\boldsymbol{k}, \tau) \right\} \,. \tag{1.42}$$

Then, after transforming the perturbations to Fourier space, we find:

$$\begin{aligned} \Psi &= \frac{1}{2k^2} \left[h'' + 6\eta'' + \mathcal{H}\left(h' + 6\eta'\right) \right] \,, \\ \Phi &= -\eta + \frac{1}{2k^2} \mathcal{H}\left(h' + 6\eta'\right) \,. \end{aligned} \tag{1.43}$$

1.2.3 Perturbed stress-energy tensor

The Einstein equations relate the Einstein tensor and the stress-energy tensor (i.e., gravity with matter). Therefore, before deriving the equations describing the evolution of the metric perturbations, we need to find the form of the linear perturbations for the stress-energy tensor.

For a perfect fluid, the stress-energy tensor is $T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}$, where ρ and P are the total proper energy density and pressure in the rest frame and $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ is the 4-velocity.[8] In locally flat coordinates in the fluid rest frame, $T^{00} = \rho$ is the energy density, $T^{i0} = 0$ is the momentum density, and $T^{ij} = P\delta^{ij}$ is the spatial stress tensor. However, an imperfect fluid may have additional components describing shear, bulk viscosity or thermal conduction. The most general stress tensor is defined as

$$T^\mu_\nu = (\rho + P)u^\mu u_\nu + Pg^\mu_\nu + \Sigma^\mu_\nu \,, \tag{1.44}$$

where $\Sigma^{\mu\nu}$ can be taken traceless and flow orthogonal ($\Sigma^\mu_\nu u^\nu = 0$). In locally flat coordinates in the fluid rest frame, only the spatial coordinates are non-zero. Under these definitions, ρu^μ is the energy-current 4-vector, including heat conduction, while P includes the bulk viscosity and $\Sigma^{\mu\nu}$ (called the shear stress), includes the shear viscosity.

This situation is very similar for a perturbed system, where we have the perturbed metric from Eq. (1.19). Let us express the perturbed stress energy

[8] Do not confuse the notation for the pressure and the 4-momentum. The latter will always have an index.

tensor as $T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu$. The total stress energy tensor is

$$\begin{aligned} T^0_0 &= -\rho\,, & T^i_0 &= -(\rho + P)v^i\,, \\ T^0_j &= (\rho + P)(w_i + v_i)\,, & T^i_j &= P\delta^i_j + \Sigma^\mu_\nu\,. \end{aligned} \tag{1.45}$$

The perturbation term is to linear order

$$\delta T^\mu_\nu = (\delta\rho + \delta P)\bar{u}^\mu \bar{u}_\nu + (\bar{\rho} + \bar{P})(\delta u^\mu \bar{u}_\nu + \bar{u}^\mu \delta u_\nu) + \delta P \delta^\mu_\nu + \Sigma^\mu_\nu\,, \tag{1.46}$$

where we are only taking into account that the shear stress is a perturbation, and we also consider the 4-velocity as its mean value plus a perturbation δu^μ. Starting from the normalization of the 4-velocity $g_{\mu\nu}u^\mu u^\nu = -1$, at linear order its perturbation is

$$\delta g_{\mu\nu}\bar{u}^\mu \bar{u}^\nu + 2\bar{u}_\mu \delta u^\mu = 0\,. \tag{1.47}$$

Since $\bar{u}^\mu = a^{-1}\delta^\mu_0$, $\bar{u}_\mu = -a\delta^0_\mu$, and $\delta g_{00} = -2a^2\Psi$, we find that $\delta u^0 = -\Psi/a$. On the other hand, δu^i is proportional to the *coordinate velocity* $v^i \equiv \mathrm{d}x^i/\mathrm{d}\tau$, finding $\delta u^i = v^i/a$. Then, at linear order,

$$u^\mu = a^{-1}\left[1 - \Psi, v^i\right]\,, \qquad u_\mu = a\left[-(1+\Psi), w_i + v_i\right]\,. \tag{1.48}$$

Substituting this expression and Eq. (1.19) in Eq. (1.46) we find at linear order

$$\begin{aligned} \delta T^0_0 &= -\delta\rho\,, & \delta T^i_0 &= -(\bar{\rho} + \bar{P})v^i\,, \\ \delta T^0_j &= (\bar{\rho} + \bar{P})(w_i + v_i)\,, & \delta T^i_j &= \delta P\delta^i_j + \Sigma^\mu_\nu\,. \end{aligned} \tag{1.49}$$

If there are several matter components, each of the quantities appearing above is the sum of all of the component contributions, except for the velocities, for which the momentum densities $(\bar{\rho} + \bar{P})v^i$ is the quantity that is additive. Finally, a similar scalar-vector-tensor decomposition can be applied to the stress-energy tensor.

In synchronous gauge, a similar approach returns $u^\mu = a^{-1}[1, v^i]$ and $u_\mu = a[-1, w_i + v_i]$ and therefore the perturbed stress energy tensor is

$$\begin{aligned} \delta T^0_0 &= -\delta\rho\,, & \delta T^i_0 &= -(\bar{\rho} + \bar{P})v^i\,, \\ \delta T^0_j &= (\bar{\rho} + \bar{P})(w_i + v_i)\,, & \delta T^i_j &= \delta P\delta^i_j + \Sigma^\mu_\nu\,. \end{aligned} \tag{1.50}$$

1.2.4 Evolution of metric perturbations

We are now ready to derive the Einstein equations at linear order. We will limit the derivation to scalar perturbations, and we will choose the Newtonian gauge (we will transform them to the synchronous gauge later using Eq. (1.32)). It is a straightforward exercise, but with very cumbersome tensor manipulations. As a reference, the metric is

$$\begin{aligned} g_{00} &= -a^2(1 + 2\Psi)\,, & g_{i0} &= 0\,, & g_{ij} &= a^2\delta_{ij}(1 + 2\Phi)\,, \\ g^{00} &= -a^{-2}(1 - 2\Psi)\,, & g^{i0} &= 0\,, & g^{ij} &= a^{-2}\delta^{ij}(1 - 2\Phi)\,, \end{aligned} \tag{1.51}$$

and the Einstein equations are

$$R^{\mu}_{\nu} - \frac{1}{2} R g^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu} \,. \tag{1.52}$$

To evaluate the left-hand side we need to compute the Christoffel symbols for the perturbed metric, use them to obtain the Ricci tensor and contract this one to form the Ricci scalar. We will work in Fourier space (changing spatial derivatives to $i\boldsymbol{k}$ factors) directly to ease the computations. We need two independent equations (one for Ψ and another one for Φ), which are easily identifiable with the 00 and scalar ij components of the Einstein equations.

1.2.4.1 Computing the pieces for the perturbed Einstein tensor

The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} (\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}). \tag{1.53}$$

The components $\Gamma^0_{\mu\nu} = -(1-2\Psi)/2a^2 \left[\partial_\mu g_{0\nu} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu}\right]$. For the 00 component the three elements in the brackets are identical, which leaves $\Gamma^0_{00} = \mathcal{H} + \Psi'$. Using a similar approach, the Christoffel symbols at linear order are

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + \Psi' \,, \\
\Gamma^0_{0i} &= i k_i \Psi \,, \\
\Gamma^0_{ij} &= \delta_{ij} \left(\mathcal{H} + 2\mathcal{H}\left[\Phi - \Psi\right] + \Phi'\right) \,, \\
\Gamma^i_{00} &= i \delta^i_j k_j \Psi \,, \\
\Gamma^i_{j0} &= \delta^i_j \left(\mathcal{H} + \Phi'\right) \,, \\
\Gamma^i_{jk} &= \left[\delta^i_j k_k + \delta^i_k k_j - \delta_{jk} \delta^i_l k_l\right] i\Phi \,.
\end{aligned} \tag{1.54}$$

The Ricci tensor is given by

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho} \,. \tag{1.55}$$

As we did before, let us explore the time-time component: $R_{00} = \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{\alpha 0}$. When $\alpha = 0$ all terms cancel directly. For the rest we have (remember that $\partial_\tau \mathcal{H} = a''/a - \mathcal{H}^2$)

$$\begin{aligned}
R_{00} &= -k^2\Psi - 3\left(\frac{a''}{a} - \mathcal{H}^2 + \Phi''\right) + 3\mathcal{H}\left(\mathcal{H} + \Psi' + \Phi'\right) - 3\mathcal{H}\left(\mathcal{H} + 2\Phi'\right) = \\
&= -k^2\Psi - 3\left(\frac{a''}{a} - \mathcal{H}^2 + \Phi''\right) + 3\mathcal{H}\left(\Psi' - \Phi'\right) \,.
\end{aligned} \tag{1.56}$$

We will skip the $0i$ component for reasons that will be apparent later, and finally the space-space part is

$$\begin{aligned}
R_{ij} = \delta_{ij} \Bigg[&\left(\frac{a''}{a} + \mathcal{H}^2\right)\left(1 + 2\Phi - 2\Psi\right) + \\
&+ \mathcal{H}\left(5\Phi' - \Psi'\right) + \Phi'' + k^2\Phi \Bigg] + k_i k_j \left(\Phi + \Psi\right) \,.
\end{aligned} \tag{1.57}$$

Now we can contract the Ricci tensor to obtain the Ricci scalar, $R \equiv g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}$, as

$$\begin{aligned} a^2R =& -(1-2\Psi)\left[-k^2\Psi - 3\left(\frac{a''}{a} - \mathcal{H}^2 + \Phi''\right) + 3\mathcal{H}\left(\Psi' - \Phi'\right)\right] + \\ &+ (1-2\Phi)\left[3\left\{\left(\frac{a''}{a} + \mathcal{H}^2\right)(1+2\Phi-2\Psi) + \right.\right. \\ &\left.\left. + \mathcal{H}\left(5\Phi' - \Psi'\right) + \Phi'' + k^2\Phi\right\} + k^2\left(\Phi+\Psi\right)\right] = \\ =& 6\frac{a''}{a} + 2k^2(\Psi + 2\Phi) + 6\Phi'' - 12\frac{a''}{a}\Psi + 6\mathcal{H}(3\Phi' - \Psi'), \end{aligned} \tag{1.58}$$

where we have separated the background and linear-order terms.

Now we have all the pieces to compute the perturbed Einstein tensor. Remember that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{1.59}$$

For the time-time component we have then

$$G_{00} = 3\mathcal{H}^2 + 6\mathcal{H}\Phi' + 2k^2\Phi, \tag{1.60}$$

where many terms cancel between the Ricci tensor and the Ricci scalar, and others are neglected due to being second order. We can skip the time-space component, since we only need two equations. For the space-space component,

$$\begin{aligned} G_{ij} = \delta_{ij}\left[-2\frac{a''}{a} + \mathcal{H}^2\right] + 2\delta_{ij}\left[2\frac{a''}{a}\left(\Psi - \Phi\right) + \mathcal{H}^2\left(\Phi - \Psi\right) + \right. \\ \left. + \mathcal{H}\left(\Psi' - 2\Phi'\right) - 2\Phi'' - k^2\left(\Phi + \Psi\right)\right] + k_ik_j\left(\Phi + \Psi\right). \end{aligned} \tag{1.61}$$

1.2.4.2 Perturbed Einstein Equations

Now we just need to manipulate the elements listed above to solve the Einstein equations. Note that substituting the elements derived in the previous subsection, we find the background solution for the Einstein Equations (i.e., the Friedman equations), hence we can cancel all background terms below, but we cannot do this directly, since some multiplicative terms may survive.

Let us consider first the trace-free part of the space-space component of the equations. The longitudinal trace-free part of the space components of a tensor A_{ij} can be obtained contracting it with $(\hat{k}^i\hat{k}^j - \delta^{ij}/3)$, therefore: $(\hat{k}^i\hat{k}^j - \frac{1}{3}\delta^{ij})\delta G_{ij} = 8\pi G(\hat{k}^i\hat{k}^j - \frac{1}{3}\delta^{ij})\delta T_{ij})$, such as

$$k^2\left(\Phi + \Psi\right) = -12\pi G a^2(\bar{\rho} + \bar{P})\sigma, \tag{1.62}$$

where we have defined $(\bar{\rho} + \bar{P})\sigma = -\left(\hat{k}_i\hat{k}_j - \delta_{ij}/3\right)\Sigma^i_j = \sum_s(\bar{\rho}_s + \bar{P}_s)\sigma_s$ as the shear (note that Σ^i_j is by definiton the traceless component of T^i_j). This is a very important result in cosmology: at linear order, and in the absence of shear, $\Psi = -\Phi$. In the standard cosmological model, there is a very small shear that is generated by photons and neutrinos, as we will see in the next section.

Now let us consider the time-time component. In this case $T^0_0 = -\rho$, and in the Newtonian gauge $T_{00} = g_{00}T^0_0 = a^2(1+2\Psi)\rho$, so that we have

$$3\mathcal{H}^2 + 6\mathcal{H}\Phi' + 2k^2\Phi = 8\pi G a^2(1+2\Psi)(\bar{\rho}+\delta\rho) = 8\pi G a^2\bar{\rho}(1+2\Psi+\delta)\,, \quad (1.63)$$

where we have defined $\delta = \delta\rho/\bar{\rho}$ and the last equality retains only linear terms. Then, canceling the background solution $3\mathcal{H}^2 = 8\pi G a^2\bar{\rho}$, we find

$$k^2\Phi + 3\mathcal{H}\left(\Phi' - \mathcal{H}\Psi\right) = 4\pi G a^2\bar{\rho}\delta\,. \quad (1.64)$$

Therefore, the two Einstein equations of interest for scalar perturbations in the Newtonian gauge are given by

$$\begin{aligned} k^2\left(\Phi+\Psi\right) &= -12\pi G a^2(\bar{\rho}+\bar{P})\sigma\,, \\ k^2\Phi + 3\mathcal{H}\left(\Phi' - \mathcal{H}\Psi\right) &= 4\pi G a^2\bar{\rho}\delta\,. \end{aligned} \quad (1.65)$$

The second equation is the generalization of the Poisson equation to an expanding and perturbed Universe. We recover the behavior of Newtonian gravity in the Newtonian regime, which is achieved for very small scales, in which expansion can be neglected, and $k \gg \mathcal{H}$.

There are two other possible equations, obtained from the $0i$ component and the trace of the spatial sector of the Einstein equation. These are

$$\begin{aligned} &-k^2\left(\Phi' - \mathcal{H}\Psi\right) = 4\pi G a^2(\bar{\rho}+\bar{P})\theta\,, \\ &-\Phi'' + \mathcal{H}(\Psi' - 2\Phi') + \left(2\frac{a''}{a} - \mathcal{H}^2\right)\Psi - \frac{k^3}{3}(\Phi+\Psi) = \frac{4\pi}{3}G a^2\delta T^i_i\,, \end{aligned} \quad (1.66)$$

where $(\bar{\rho}+\bar{P})\theta = \sum_s(\bar{\rho}_s + \bar{P}_s)\theta_s$ and $\theta = ik_iv^i$ is the divergence of the coordinate (or fluid) velocity.

1.2.4.3 Einstein equations in synchronous gauge

Using the equivalence between the metric perturbations in Newtonian and synchronous gauge from Eq. (1.43), we can transform the Einstein equations above and express them in synchronous gauge. To do so, we also need to transform the stress energy tensor between the Newtonian and the synchronous gauge. The transformation is given by

$$T^\mu_\nu(\text{Syn}) = \frac{\partial\hat{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\hat{x}^\nu}T^\alpha_\beta(\text{Newt})\,, \quad (1.67)$$

where as above the hat coordinates are in synchronous gauge. With this procedure we find that $T^0_0(\text{Syn}) = T^0_0(\text{Newt})$, $T^0_j(\text{Syn}) = T^j_0(\text{Newt}) + ik_j\alpha(\bar{\rho}+\bar{P})$, and $T^j_i(\text{Syn}) = T^j_i(\text{Newt})$, where $\alpha = (h' + 6\eta')/2k^2$. For the quantities of interest, this implies that evaluating the perturbations at the same spacetime coordinates, we have $\delta_\mathrm{S} = \delta_\mathrm{N} - \alpha\bar{\rho}'/\bar{\rho}$, $\theta_\mathrm{S} = \theta_\mathrm{N} - \alpha k^2$, $(\delta p)_\mathrm{S} = (\delta p)_\mathrm{N} - \alpha\bar{P}'$ and $\sigma_\mathrm{S} = \sigma_\mathrm{N}$, where the subscripts denote synchronous and Newtonian gauges. Then, we have

$$\begin{aligned} & h'' + 6\eta'' + 2\mathcal{H}(h' + 6\eta'') - 2k^2\eta = -24\pi G a^2(\bar{\rho} + \bar{P})\sigma\,, \\ & -k^2\eta + \frac{1}{2}\mathcal{H}h' = 4\pi G a^2\bar{\rho}\delta_\mathrm{S}\,. \end{aligned} \tag{1.68}$$

And the other two equations are given by

$$\begin{aligned} & k^2\eta' = 4\pi G a^2(\bar{\rho} + \bar{P})\theta_\mathrm{S}\,, \\ & h'' + 2\mathcal{H}h' - 2k^2\eta = -8\pi G a^2\delta T^i_i\,. \end{aligned} \tag{1.69}$$

Note that if the density is defined in the synchronous gauge, the standard Newtonian Poisson equation holds in all scales. This can be seen substituting Φ in the equation above using the equivalence in Eq. (1.43) and accounting for the relation between the Newtonian and synchronous determinations of the matter perturbation (see above).

1.3 Perturbed Boltzmann Equations

At the beginning of this chapter we discussed the Boltzmann equations in a smooth expanding Universe. Now it is time to introduce metric perturbations in the formalism. Metric perturbations affect how particles move, which in turn affect the phase-space distribution. We need to know how the position, momentum and direction of the momentum change with time. Of course, the actual expressions depend on the gauge.

The mass-shell constrain $g_{\mu\nu}P^\mu P^\nu = -m^2$ is now given by

$$\begin{aligned} & \mathrm{Newt}: a^2(1+2\Psi)(P^0)^2 + p^2 = -m^2 \\ & \mathrm{Syn}: a^2(P^0)^2 + p^2 = -m^2\,, \end{aligned} \tag{1.70}$$

where as always $p^2 = g_{ij}P^iP^j$. Defining still the energy as in the unperturbed case, $E = \sqrt{p^2 + m^2}$, the time component of P^μ is determined by the energy and the metric perturbation. At linear order,

$$\begin{aligned} & \mathrm{Newt}: P^\mu = [E(1-\Psi)/a, p_i(1-\Phi)/a] \\ & \mathrm{Syn}: P^\mu = \left[E/a, (\delta_{ij} - \frac{1}{2}h_{ij})p^j/a\right]\,, \end{aligned} \tag{1.71}$$

where we have left everything at linear order (e.g., note the factors $(1 + 2A)^{-1/2} \to (1-A)$). As derived above, in the absence of metric perturbations, Hamilton's equations state that the conjugate momentum (which is the spatial part of the 4-momentum with lower indices, $g_{ij}P^j$) must be constant; hence, the proper momentum $p_i = p^i$ must redshift as a^{-1}. Then, the conjugate momenta is $P_i = ap_i(1+\Phi)$ in Newtonian gauge and $P_i = a(\delta_{ij} + \frac{1}{2}h_{ij})p^j$ in synchronous gauge.

Note that the phase-space distribution is a scalar and is invariant under canonical transformations. Its zeroth-order is either a Fermi-Dirac (+) or a Bose-Einstein (−) distribution:

$$f_0 = f_0(\epsilon) = \frac{g_*}{h_P^3} \frac{1}{\exp\{\epsilon/k_B T_0\} \pm 1}\,, \tag{1.72}$$

where we have defined $\epsilon = aE = a\sqrt{p^2+m^2}$ and $T_0 = aT$ as the temperature of the particles today for convenience, and h_P and k_B are the Planck and the Boltzmann constants, respectively. ϵ is related to the time component of the 4-momentum by $P_0 = -\epsilon$ in the synchronous gauge and $P_0 = -\epsilon(1+\Psi)$. Note also that now we write the degeneracy factor in the actual distribution function, contrary as before.

Also for convenience, let us replace the conjugate momentum P_i by the comoving momentum $q_i \equiv ap_i$ in order to eliminate the metric perturbations from the definition of the momenta, and as we have done before we separate $q_i = q\hat{q}_i$ on its magnitude and direction. Then, $f(x^i, P_j, \tau) \to f(x^i, q, \hat{q}_j, \tau)$. Note that q_j is not the conjugate momentum and we cannot consider the phase-space volume element to be $\mathrm{d}^3\boldsymbol{x}\mathrm{d}\boldsymbol{q}/(2\pi)^3$. In practice we have moved the impact of the perturbations from the variable to the phase-space volume element: for instance the number of particles is

$$\mathrm{d}N = \frac{f\mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3} = (1+3\Phi)\frac{f\mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\,, \tag{1.73}$$

which is reasonable since the proper distance is $a(1+\Phi)\mathrm{d}x^i$.

Remember that the general expression for the stress-energy tensor can be written as

$$T_{\mu\nu} = \frac{1}{\sqrt{-|g_{\alpha\beta}|}} \int \mathrm{d}^3\boldsymbol{P} \frac{P_\mu P_\nu}{P^0} f(x^i, P_j, \tau)\,, \tag{1.74}$$

and, as done with all quantities so far, we can treat the distribution function perturbatively,

$$f(x^i, P_j, \tau) = f_0(q, m)\left(1 + \varphi(x^i, q, \hat{q}_j, \tau)\right)\,, \tag{1.75}$$

such as the only thing left is to transform the geometric factors from Eq. (1.74).

In synchronous gauge, $(-|g_{\alpha\beta}|)^{-1/2} = [a^8(1+h/3)^3]^{-1/2}$, which at linear order is $(-|g_{\alpha\beta}|)^{-1/2} = (1-h/2)/a^4$. Similarly, $\mathrm{d}^3\boldsymbol{P} = (1+h/2)q^2\mathrm{d}q\mathrm{d}\Omega_q$, where Ω_q is the solid angle for $\hat{q}_j$. Note that the factor $(1+h/2)(1-h/2) = 1$ at linear order. Now we can express the components of the stress-energy tensor in terms of the perturbed phase-space distribution (substituting the

4-momenta in Eq. (1.74)):

$$
\begin{aligned}
T^0_0 &= -a^{-4}\int \mathrm{d}q\mathrm{d}\Omega_\mathrm{q} q^2\sqrt{q^2+m^2a^2} f_0(q,m)(1+\varphi)\,,\\
T^0_i &= a^{-4}\int \mathrm{d}q\mathrm{d}\Omega_\mathrm{q} q^2 q\hat{q}_i f_0(q,m)\varphi\,,\\
T^i_j &= a^{-4}\int \mathrm{d}q\mathrm{d}\Omega_\mathrm{q} q^2 \frac{q^2\hat{q}^i\hat{q}_j}{\sqrt{q^2+m^2a^2}} f_0(q,m)(1+\varphi)\,,
\end{aligned}
\tag{1.76}
$$

where we have used that $\int \mathrm{d}\Omega_\mathrm{q}\hat{q}_i = \int \mathrm{d}\Omega_\mathrm{q}\hat{q}_i\hat{q}_j\hat{q}_k = 0$ (which cancels the unperturbed f_0 term in T^0_i) and $\int \mathrm{d}\Omega_\mathrm{q}\hat{q}_i\hat{q}_j = 4\pi\delta_{ij}/3$ (which makes that the term in T^i_j survives).

Similarly in the Newtonian gauge, $(-|g_{\alpha\beta}|)^{-1/2} = (1-\Psi-3\Phi)/a^4$ and, as said above, $\mathrm{d}^3\boldsymbol{P} = (1+3\Phi)q^2\mathrm{d}q\mathrm{d}\Omega_\mathrm{q}$. Applying the same procedure, we find that the expression for the stress-energy tensor is the same in Newtonian gauge, although with q_j referring in this case to the 4-momentum in Newtonian gauge.

The general Boltzmann equation is, in terms of the variables discussed now,

$$
\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}\tau} + \frac{\partial f}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\tau} + \frac{\partial f}{\partial \hat{q}_i}\frac{\mathrm{d}\hat{q}_i}{\mathrm{d}\tau} = C[f]\,. \tag{1.77}
$$

Then, we need to obtain the total derivatives as function of τ to obtain the expression in each gauge. The total derivatives (how position and momentum change with time in the absence of collisions) is where gravity (through the determination of the geodesics) chimes in. Remember that $P^i \equiv \mathrm{d}x^i/\mathrm{d}\lambda$ and $P^0 \equiv \mathrm{d}\tau/\mathrm{d}\lambda$, such as at linear order

$$
\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \frac{P^i}{P^0}\,, \tag{1.78}
$$

which corresponds to

$$
\begin{aligned}
\text{Newt}:\ \frac{\mathrm{d}x^i}{\mathrm{d}\tau} &= q\hat{q}^i(1-\Phi+\Psi)/\epsilon\,,\\
\text{Syn}:\ \frac{\mathrm{d}x^i}{\mathrm{d}\tau} &= q(\hat{q}^i + h_{ij}\hat{q}^j)/\epsilon\,,
\end{aligned}
\tag{1.79}
$$

although since this term multiplies the gradient of f and f_0 is homogeneous, we can neglect all perturbations in the expression above, since φ is already a linear-order term.

Remember the geodesic equation

$$
\frac{\mathrm{d}P^\mu}{\mathrm{d}\lambda} = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta\,. \tag{1.80}
$$

Using that $\mathrm{d}/\mathrm{d}\lambda = (\mathrm{d}x^\mu/\mathrm{d}\lambda)(\mathrm{d}/\mathrm{d}x^\mu) = P^\mu\mathrm{d}/\mathrm{d}x^\mu$, we have for the space component,

$$
P^0\frac{\mathrm{d}P^i}{\mathrm{d}\tau} + P^j\frac{\mathrm{d}P^i}{\mathrm{d}x^j} = -\Gamma^i_{\alpha\beta}P^\alpha P^\beta\,. \tag{1.81}
$$

Now we can straightforwardly obtain the derivative with respect q^i doing some algebra: we start from P^i and go to p^i, and from this to p and q. Then, we need to use Eq. (1.71) in the equation above and propagate. We will go through the derivation for the Newtonian gauge. To start, note that

$$\begin{aligned} \frac{\mathrm{d}P^i}{\mathrm{d}\tau} &= \frac{1-\Phi}{a}\frac{\mathrm{d}p^i}{\mathrm{d}\tau} - \frac{p^i}{a}\left(\mathcal{H}[1-\Phi]+\Phi'\right) , \\ \frac{\mathrm{d}P^i}{\mathrm{d}x^j} &= -\frac{ip^i k_j \Phi}{a} . \end{aligned} \tag{1.82}$$

Substituting this in Eq. (1.81) and isolating $\mathrm{d}p^i/\mathrm{d}\tau$, we find that the geodesic equation is transformed to

$$\begin{aligned} \frac{\mathrm{d}p^i}{\mathrm{d}\tau} =& \frac{a^2(1+\Phi+\Psi)}{E}\times \\ &\times \left\{ -\Gamma^i_{\alpha\beta}P^\alpha P^\beta + \frac{ip^ip^jk_j\Phi}{a^2} + \frac{Ep^i}{a^2}\left(\mathcal{H}[1-\Phi-\Psi]+\Phi'\right)\right\} . \end{aligned} \tag{1.83}$$

We have to compute now the term with the Christoffel symbols, using Eq. (1.54). Expanding the expression we have

$$-\Gamma^i_{\alpha\beta} = -\left(\Gamma^i_{00}P^0P^0 + 2\Gamma^i_{0j}P^0P^j + \Gamma^i_{jk}P^jP^k\right) . \tag{1.84}$$

Note that the Christoffel symbols in the first and last terms are already first order, so we can ignore the contributions from the perturbations in the momenta. Neglecting higher-order terms, we have

$$\begin{aligned} -\Gamma^i_{\alpha\beta} &= -i\frac{E^2}{a^2}k_i\Psi - \\ &- \frac{2Ep^i}{a^2}\left(\mathcal{H}[1-\Psi-\Phi]+\Phi'\right) - \\ &- \frac{i\Phi}{a^2}\left(p^ip_kk_k + p^ip_kk_k - p^2k_i\right) . \end{aligned} \tag{1.85}$$

Therefore, adding all contributions, we have

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = -p^i(\mathcal{H}+\Phi') - iEk_i\Psi - i\frac{\Phi}{E}(p^ip^jk_j - p^2k_i) , \tag{1.86}$$

and using that

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\delta_{ij}p^ip^j} = \delta_{ij}\frac{p^i}{p}\frac{\mathrm{d}p^j}{\mathrm{d}\tau} , \tag{1.87}$$

we have

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = -p(\mathcal{H}+\Phi') - iE\hat{p}^ik_i\Psi . \tag{1.88}$$

However the cumbersome calculation, it would have been possible to qualitatively guess the result, since the first term corresponds to the loss of momentum due to the Hubble expansion (including cosmological redshift and decay of the peculiar velocity) and the second term encodes the effect of the particle traveling into a potential well. The last two terms in Eq. (1.86) cancel when taking the norm, and this is because they do not change the particle's momentum at linear order, but they do change its direction.

The expression above in the variables we want to use, $q = ap$ and $\epsilon = aE$, converts to

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = -q\Phi' - i\epsilon\hat{q}^i k_i \Psi\,. \tag{1.89}$$

The equivalent expression in synchronous gauge is

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = -\frac{1}{2} q h'_{ij}\hat{q}_i\hat{q}_j\,. \tag{1.90}$$

Finally, since $\partial f/\partial\hat{q}$ is also a first-order quantity, the last term in the left-hand side of Eq. (1.77) can be neglected to first order. Joining all the terms and keeping only first-order quantities, the perturbed boltzmann equation, i.e., the evolution of the perturbation φ of the phase-space distribution ($f = f_0(1+\varphi)$), is given by

$$\begin{aligned} \text{Newt}: \frac{\partial\varphi}{\partial\tau} + i\frac{q}{\epsilon}\boldsymbol{k}\hat{\boldsymbol{q}}\varphi + \frac{\partial\log f_0}{\partial\log q}\left(-\Phi' - i\frac{\epsilon}{q}\boldsymbol{k}\hat{\boldsymbol{q}}\Psi\right) = \frac{C[f]}{f_0}\,, \\ \text{Syn}: \frac{\partial\varphi}{\partial\tau} + i\frac{q}{\epsilon}\boldsymbol{k}\hat{\boldsymbol{q}}\varphi + \frac{\partial\log f_0}{\partial\log q}\left(\eta' - \frac{h'+6\eta'}{2}(\hat{\boldsymbol{k}}\hat{\boldsymbol{q}})^2\right) = \frac{C[f]}{f_0}\,. \end{aligned} \tag{1.91}$$

As can be seen above, the Boltzmann equation only depends on the direction of the momentum through its angle with $\boldsymbol{k}$ (further dependence can be introduced in the collision term). Therefore if the momentum dependence of the initial phase-space perturbation is axially symmetric about $\boldsymbol{k}$, it will remain axially symmetric. This implies that if axially-asymmetric perturbation in the neutrinos or any other collisionless particles are produced, they would generate no scalar perturbation and therefore would have no effect on other species. We will take this assumption, so that φ only depends on $\hat{\boldsymbol{q}}$ through the product $\hat{\boldsymbol{k}}\hat{\boldsymbol{q}}$.

In some cases, it will also be useful to keep the perturbed Boltzmann equation for the whole distribution (assuming that the zero-th order is homogeneous and does not depend on the direction of the momentum), and using the proper momentum p and energy E, rather than q and ϵ. Note that in this case we only need to change the variables from Eq. (1.77), with the subtlety that $\mathrm{d}q/\mathrm{d}\tau = a(\mathrm{d}p/\mathrm{d}\tau + \mathcal{H}p)$. Then, we have

$$\begin{aligned} \text{Newt}: \frac{\partial f}{\partial\tau} + i\frac{p}{E}\boldsymbol{k}\hat{\boldsymbol{p}}f + \frac{\partial f}{\partial p}p\left(-\mathcal{H} - \Phi' - i\frac{E}{p}\boldsymbol{k}\hat{\boldsymbol{p}}\Psi\right) = C[f]\,, \\ \text{Syn}: \frac{\partial f}{\partial\tau} + i\frac{p}{E}\boldsymbol{k}\hat{\boldsymbol{p}}f + \frac{\partial f}{\partial p}p\left(-\mathcal{H} + \eta' - \frac{h'+6\eta'}{2}(\hat{\boldsymbol{k}}\hat{\boldsymbol{p}})^2\right) = C[f]\,. \end{aligned} \tag{1.92}$$

1.4 Evolution of matter perturbations

Now we have all the tools to compute the perturbations of all components in the Universe that contribute to the stress-energy tensor.[9] We will consider cold dark matter, baryons, massless and massive neutrinos and photons. In general, we will make derivations in Newtonian gauge and provide the expressions in synchronous gauge for completeness.

1.4.1 Dark matter

Dark matter makes up for most of the non-relativistic matter in the Universe, and it is mostly cold. We will consider a completely collisionless cold dark matter, i.e., the dark matter does not interact with any other species in the Universe or itself other than gravitationally and it is completely non relativistic. This means $C[f]=0$, and that factors $q/\epsilon = p/E \sim p/m$ will be very small: we will only retain up to linear-order terms in p/m, which accounts for the bulk motion of dark matter but not its velocity dispersion. These assumptions make that dark matter can be treated as a pressure-less effective fluid which is described by its density and velocity. We will derive the evolution equations taking moments of the Boltzmann equations.

From the phase-space distribution we can take the description of a collection of particles if we integrate over phase-space volume elements. Remember that the number density and the fluid velocity can be obtained by integrating over the proper momentum; denoting dark matter with a subscript 'c',

$$n_c = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} f_c\,, \qquad n_c v_c^i = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{p\hat{p}^i}{E} f_c\,. \tag{1.93}$$

Then, if we multiply the Boltzmann equation in Eq. (1.92) for the whole distribution by the phase-space element and integrate we have

$$\begin{aligned}\frac{\partial}{\partial\tau}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} f_c + i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{p\boldsymbol{k}\hat{\boldsymbol{p}}}{E} f_c - (\mathcal{H}+\Phi')\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial f_c}{\partial p}p - \\ - \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial f_c}{\partial p} iE\boldsymbol{k}\hat{\boldsymbol{p}}\Psi = 0\,,\end{aligned} \tag{1.94}$$

where the time derivative can be place outside of the integral in the first term because $\mathbf{p}$ is an independent variable here, we can substitute the first two terms by the number density and fluid velocity and integrating by parts the integral in the third term is

$$\frac{1}{(2\pi)^3}\int \mathrm{d}pp^3\frac{\partial}{\partial p}\int \mathrm{d}\Omega_{\mathrm{p}} f_c = \frac{-3}{(2\pi)^3}\int \mathrm{d}pp^2\frac{\partial}{\partial p}\int \mathrm{d}\Omega_{\mathrm{p}} f_c = -3n_c\,, \tag{1.95}$$

[9]For effective fuilds, the fluid equations can be obtained by taking moments in q of the Boltzmann equations, similar to Eq. (1.74).

and the fourth vanishes. Then, the zero-th moment of the Boltzmann equation returns

$$\frac{\partial n_c}{\partial \tau} + i n_c \boldsymbol{k}\boldsymbol{v}_c + 3(\mathcal{H} + \Phi')n_c = 0\,, \tag{1.96}$$

which is the cosmological generalization of the continuity equation, including the last term to account for the perturbations of the metric and the expansion of the Universe. The zero-th order in perturbations above returns (remember that the velocity is already a first-order perturbation)

$$\frac{\partial \bar{n}_c}{\partial \tau} + 3\mathcal{H}\bar{n}_c = 0\,, \tag{1.97}$$

which shows that $n_c \propto a^{-3}$ for non-relativistic matter, as discussed before in the course. Perturbing this number density as $n_c = \bar{n}_c(1+\delta_c)$ (which also fulfills previous definitions of δ), and dividing by $a^3\bar{n}_c$ we find

$$\delta_c' = -\theta_c - 3\Phi'\,, \tag{1.98}$$

where we have recovered the definition of θ as the velocity divergence. We still need another equation to determine the evolution of δ_c and θ_c, which we can get by using the first moment of the Boltzmann equation (weighting the integral with $p\hat{p}^j/E$):

$$\frac{\partial}{\partial \tau}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{p\hat{p}^j}{E}f_c + i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{p^2\hat{p}^j\boldsymbol{k}\hat{\boldsymbol{p}}}{E^2}f_c - (\mathcal{H}+\Phi')\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial f_c}{\partial p}\frac{p^2\hat{p}^j}{E} - \\ - \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial f_c}{\partial p}p\hat{p}^j i\boldsymbol{k}\hat{\boldsymbol{p}}\Psi = 0\,. \tag{1.99}$$

The first term is the time derivative of $a^3 n_c v_c^j$ and the second can be neglected, since it is second order in p/E. Integrating by parts the third term we get

$$\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial f_c}{\partial p}\frac{p^2\hat{p}^j}{E} = \int \frac{\mathrm{d}\Omega_\mathrm{p}}{(2\pi)^3}\hat{p}^j\int \mathrm{d}p\frac{p^4}{E}\frac{\partial f_c}{\partial p} = \\ = -\int \frac{\mathrm{d}\Omega_\mathrm{p}}{(2\pi)^3}\hat{p}^j\int \mathrm{d}p f_c\left(\frac{4p^3}{E} - \frac{p^5}{E^3}\right)\,. \tag{1.100}$$

the first term in the brackets yields $-4a^3 n_c v_c^j$, while the second term is higher order in p/E and thus can be neglected. Applying the same approach to the last term in the weighted integral of the Boltzmann equation and considering that $\int \mathrm{d}\Omega_\mathrm{p}\hat{p}^i\hat{p}^j = \delta^{ij}4\pi/3$, we get that the first moment of the Boltzmann equation is

$$\frac{\partial(n_c v_c^j)}{\partial \tau} + 4\mathcal{H}n_c v_c^j + i n_c k^j\Psi = 0\,. \tag{1.101}$$

Since all terms are already first order, we can directly write n_c as $\bar{n}_c$, and use its background evolution. After multiplying by ik_j, we find

$$\theta_c' = -\mathcal{H}\theta_c + k^2\Psi\,. \tag{1.102}$$

This is the momentum conservation, or Euler equation, although in this case it does not contain the standard $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ term because it is second order. Even if the dark matter perturbations are the simplest ones, we can already see a common feature of integrating the Boltzmann equations: the integrated n-th moment always depends on the $(n+1)$-th moment. This leads to an infinite hierarchy of equations that needs to be closed at some moment. In the case of cold dark matter the hierarchy is closed setting the second moment to zero (which follows from the assumption that the dark matter is cold) and the drop of $(p/E)^2$ and higher terms. This will not be possible for relativistic species as neutrinos and photons, as we will see below.

To summarize, the Boltzmann equations for dark matter in Newtonian gauge are

$$\delta_c' = -\theta_c - 3\Phi' \,, \qquad \theta_c' = -\mathcal{H}\theta_c + k^2\Psi \,, \tag{1.103}$$

and the equivalent in synchronous gauge,

$$\delta_c' = -\frac{1}{2}h' \,, \qquad \theta_c = 0 \,, \tag{1.104}$$

where the $\theta_c = 0$ condition, i.e., that cold dark matter particules have zero peculiar velocities in the synchronous gauge, is used to define the synchronous coordinates.

Dark matter perturbations are the simplest ones. The next step in complication would be the baryon perturbations, but since they are coupled to photons through Compton scattering, we need to derive the collision term. Since that is an integral part of the derivation of the Boltzmann equation for photons, we leave the discussion of baryons for later, and proceed first with the neutrino perturbations.

1.4.2 Massless neutrinos

The energy density and pressure for massless neutrinos (labeled with subscript 'ν') are $\rho_\nu = 3P_\nu = -T^0_0 = T^i_i$. From Eq. (1.76), we can find the background energy density and pressure are $\bar{\rho}_\nu = 3\bar{P}_\nu = a^{-4}\int \mathrm{d}q\mathrm{d}\Omega_\mathrm{q} q^2 q f_0(q)$, and that their perturbations and the energy flux $\delta T^0_{\nu i}$ and shear stress $\Sigma^i_{\nu j} = T^i_{\nu j} - P_\nu\delta^i_j$ (the background energy flux and shear stress are zero) are

$$\begin{aligned}
\delta\rho_\nu = 3\delta P_\nu = a^{-4}\int q^2\mathrm{d}q\mathrm{d}\Omega_\mathrm{q} q f_0(q)\varphi \,, \\
\delta T^0_{\nu i} = a^{-4}\int q^2\mathrm{d}q\mathrm{d}\Omega_\mathrm{q} q\hat{q}^i f_0(q)\varphi \,, \\
\Sigma^i_{\nu j} = a^{-4}\int q^2\mathrm{d}q\mathrm{d}\Omega_\mathrm{q} q(\hat{q}^i\hat{q}_j - \frac{1}{3}\delta^i_j) f_0(q)\varphi \,.
\end{aligned} \tag{1.105}$$

Note also that, for massless particles, $q = \epsilon$. Since the quantities involved in the stress-energy tensor are weighted integrals of the phase space, we can further reduce the number of variables if we integrate out the q-dependence of φ and expand the angular dependence in a series of Legendre polynomials $\mathcal{P}_\ell(\mu)$, where $\mu = \hat{\boldsymbol{k}}\hat{\boldsymbol{q}}$. Let us define

$$\mathcal{F}_\nu(\boldsymbol{k}, \hat{q}, \tau) \equiv \frac{\int q^2 \mathrm{d}q q f_0 \varphi}{\int q^2 \mathrm{d}q q f_0} \equiv \sum (-i)^\ell (2\ell+1) \mathcal{F}_{\nu\ell}(\boldsymbol{k}, \tau) \mathcal{P}_\ell(\mu) \,, \tag{1.106}$$

where the factor $(-i)^\ell(2\ell+1)$ has been chosen to simplify the expansion of a plane wave: $\mathcal{F}_\nu = \exp(-ikr\mu)$ has expansion coefficients $\mathcal{F}_{\nu\ell} = j_\ell(kr)$ given by the spherical Bessel functions. The purpose of the expansion in Legendre polynomials is to remove the explicit dependence in μ, which complicates the computations.

The fluid variables of interest ($\delta \equiv \delta\rho/\bar{\rho}$, $\theta \equiv ik^j \delta T^0_j/(\bar{\rho} + \bar{P})$, and $\sigma \equiv -(\hat{k}_i \hat{k}^j - \delta^j_i/3)\Sigma^i_j/(\bar{\rho} + \bar{P})$) can be expressed in terms of the expansion coefficients of the new variable $\mathcal{F}_\nu$ by performing the corresponding weighted angular integral to $\mathcal{F}_\nu$. From Eq. (1.105):

$$\begin{aligned}
\delta_\nu &= \frac{1}{4\pi} \int \mathrm{d}\Omega_{\mathrm{q}} \mathcal{F}_\nu = \mathcal{F}_{\nu 0} \,, \\
\theta_\nu &= \frac{3i}{16\pi} \int \mathrm{d}\Omega_{\mathrm{q}} (\hat{\boldsymbol{k}}\hat{\boldsymbol{q}}) k \mathcal{F}_\nu = \frac{3}{4} k \mathcal{F}_{\nu 1} \,, \\
\sigma_\nu &= -\frac{3}{16\pi} \int \mathrm{d}\Omega_{\mathrm{q}} \left[(\hat{\boldsymbol{k}}\hat{\boldsymbol{q}})^2 - \frac{1}{3} \right] \mathcal{F}_\nu = \frac{1}{2} \mathcal{F}_{\nu 2} \,,
\end{aligned} \tag{1.107}$$

where the division of the background quantities is already in the denominator of the definition of $\mathcal{F}_\nu$ and the numerical prefactors account for the angular integrals of the background, homogeneous distribution function and the match with the Legendre coefficients.

Integrating Eq. (1.91) over $q^2 \mathrm{d}q f_0$ and dividing them by $\int q^2 \mathrm{d}q f_0$ (i.e., applying the definition of $\mathcal{F}_\nu$ above to the evolution of the perturbation φ of the phase-space distribution function and using $q = \epsilon$), the Boltzmann equation for massless neutrinos becomes

$$\begin{aligned}
\text{Newt}: \mathcal{F}'_\nu &= -ik\mu \mathcal{F}_\nu - 4(\Phi' + ik\mu\Psi) \,, \\
\text{Syn}: \mathcal{F}'_\nu &= -ik\mu \mathcal{F}_\nu - \frac{2}{3} \left[h' + 2(h' + 6\eta') \mathcal{P}_2(\mu) \right] \,,
\end{aligned} \tag{1.108}$$

where $\left[\int q^2 \mathrm{d}q f_0 \mathrm{d} \log f_0 / \mathrm{d} \log q \right] / \int q^2 \mathrm{d}q f_0 = -4$, $\mathcal{P}_2(\mu) = (3\mu^2 - 1)/2$ and the expression of the last two terms in the synchronous gauge has been treated to be expressed in terms of $\mathcal{P}_2$.

Now we can mix Eqs. (1.108) and (1.106) to obtain the evolution for the coefficients. For instance, in the Newtonian gauge,

$$\begin{aligned}
\sum (-i)^\ell (2\ell+1) \mathcal{F}'_{\nu\ell} \mathcal{P}_\ell(\mu) = &- k \sum (-i)^{\ell+1} (2\ell+1) \mathcal{F}_{\nu\ell} \mu \mathcal{P}_\ell(\mu) \\
&- 4(\Phi' + ik\mu\Psi) \,.
\end{aligned} \tag{1.109}$$

We can use the orthonormality of the Legendre polynomials and the recursion relation of $(2\ell+1)\mu\mathcal{P}_\ell(\mu) = \ell\mathcal{P}_{\ell-1}(\mu) + (\ell+1)\mathcal{P}_{\ell+1}(\mu)$, such as if we multiply each side of the equation above by $\mathcal{P}_{\ell'}$ and integrate over μ we can get the relations that we need. For instance, if $\ell' = 0$ we can do the whole sum and only few terms survive (and noting that $\mathcal{P}_1 \propto \mu$):

$$\begin{aligned} \delta_\nu' &= -\frac{4}{3}\theta_\nu - 4\Phi' \,, \\ \theta_\nu' &= k^2 \left(\frac{1}{4}\delta_\nu - \sigma_\nu\right) + k^2\Psi \,, \\ \mathcal{F}_{\nu\ell}' &= \frac{k}{2\ell+1}\left[\ell\mathcal{F}_{\nu\ell-1} - (\ell+1)\mathcal{F}_{\nu\ell+1}\right] \,, \qquad \ell \geq 2 \,, \end{aligned} \tag{1.110}$$

where we have applied the procedure to all multipoles and have matched the remaining terms to the μ dependence. And for the synchronous gauge:

$$\begin{aligned} \delta_\nu' &= -\frac{4}{3}\theta_\nu - \frac{2}{3}h' \,, \\ \theta_\nu' &= k^2 \left(\frac{1}{4}\delta_\nu - \sigma_\nu\right) \,, \\ \mathcal{F}_{\nu 2}' &= 2\sigma_\nu' = \frac{8}{15}\theta_\nu - \frac{3}{5}k\mathcal{F}_{\nu 3} + \frac{4}{15}h' + \frac{8}{5}\eta' \,, \\ \mathcal{F}_{\nu\ell}' &= \frac{k}{2\ell+1}\left[\ell\mathcal{F}_{\nu\ell-1} - (\ell+1)\mathcal{F}_{\nu\ell+1}\right] \,, \qquad \ell \geq 3 \,. \end{aligned} \tag{1.111}$$

where in the synchronous gauge we need to specify the $\ell = 2$ case because of the μ dependence in h' and η' in the $\mathcal{F}_\nu'$ equation (the presence of $\mathcal{P}_2$).

Note that for a given ℓ, $\mathcal{F}_{\nu\ell}$ is coupled to the two neighbouring modes, and that a priori the Boltzann hierarchy is infinite. Therefore, we need to truncate at some $\ell_{\rm max}$. One option is to set $\mathcal{F}_{\nu\ell} = 0$ for $\ell > \ell_{\rm max}$, but this is inaccurate because the error in the coupling at $\ell_{\rm max}$ propagates to smaller ℓ due to the coupling between modes.[10] An improved truncation scheme is based in the extrapolation of the behavior of $\mathcal{F}_{\nu\ell}$ at $\ell = \ell_{\rm max} + 1$. More sophisticated schemes have been developed to improve the accuracy of the Boltzmann equations, including an exact solution transforming Eq. (1.108) into an integral equation, which allows to solve the system iteratively.

1.4.3 Massive neutrinos

Massive neutrinos are a very particular species in the Universe. Their mass, which sums to $0.06\,\mathrm{eV} \leq \sum m_\nu \lesssim 0.1$ eV implies that they were relativistic particles until $z \sim 100$, when they become non relativistic as the Universe expands and they get colder. They can be considered hot dark matter, and we will denote them with the subscript 'h'. The evolution of the perturbations

[10]The error propagates to $\ell = 0$ in a time $\tau \approx \ell_{\rm max}/k$ and the reflects back to increasing ℓ, due again to the coupling, increasing the errors even more.

to their distribution function is more complicated than in the case of massless neutrinos due to the finite mass.

Experimental and observational evidence cannot distinguish between the normal and the inverted hierarchy yet, and cannot determine whether any of the neutrinos is effectively massless or not. However, cosmological perturbations are practically sensitive only to the total neutrino mass, not able to distinguish between individual neutrino masses. Since the evolution of massless neutrinos is significantly simpler (and cheaper to compute), it is customary to consider a single massive neutrino and 2 massless neutrinos in the set of Boltzmann equations.

In this case, we cannot ignore the neutrino mass (i.e., $q \neq \epsilon = (q^2 + m_\nu^2 a^2)^{1/2}$). From Eqs. (1.76) and (1.105), the unperturbed energy density and pressure, and the corresponding perturbations, are

$$\begin{aligned} \bar{\rho}_h &= a^{-4}\int q^2 \mathrm{d}q \mathrm{d}\Omega_{\mathrm{q}} \epsilon f_0\,, \quad \bar{P}_h = \frac{1}{3}a^{-4}\int q^2 \mathrm{d}q \mathrm{d}\Omega_{\mathrm{q}} \frac{q^2}{\epsilon} f_0 \varphi\,, \\ \delta\rho_h &= a^{-4}\int q^2 \mathrm{d}q \mathrm{d}\Omega_{\mathrm{q}} \epsilon f_0 \varphi\,, \quad \delta P_h = \frac{1}{3}a^{-4}\int q^2 \mathrm{d}q \mathrm{d}\Omega_{\mathrm{q}} \frac{q^2}{\epsilon} f_0 \varphi\,, \\ \delta T^0_{hi} &= a^{-4}\int q^2 \mathrm{d}q \mathrm{d}\Omega_{\mathrm{q}} q \hat{q}_i f_0\,, \quad \Sigma^i_{hj} = a^{-4}\int q^2 \mathrm{d}q \mathrm{d}\Omega_{\mathrm{q}} \frac{q^2}{\epsilon}\left(\hat{q}^i \hat{q}_j - \frac{1}{3}\delta^i_j\right) f_0 \varphi\,. \end{aligned} \tag{1.112}$$

Now we can proceed with the same philosophy as for the massless neutrinos, but note that here there is a critical difference. The energy-momentum relation depends *both* in time and the momentum, which prevents us to integrate out the q dependence as we did before. This forces us to expand φ in the Legendre polynomial series directly:

$$\varphi(\boldsymbol{k}, \hat{\boldsymbol{q}}, q, \tau) = \sum (-i)^\ell (2\ell+1)\varphi_\ell(\boldsymbol{k}, q, \tau)\mathcal{P}_\ell(\mu)\,, \tag{1.113}$$

which, after integration over the angular variables, leaves the perturbations of interest as

$$\begin{aligned} \delta\rho_h &= 4\pi a^{-4}\int q^2 \mathrm{d}q \epsilon f_0 \varphi_0\,, \\ \delta P_h &= \frac{4\pi}{3}a^{-4}\int q^2 \mathrm{d}q \frac{q^2}{\epsilon} f_0 \varphi_0\,, \\ (\bar{\rho}_h + \bar{P}_h)\theta_h &= 4\pi k a^{-4}\int q^2 \mathrm{d}q q f_0 \varphi_1\,, \\ (\bar{\rho}_h + \bar{P}_h)\sigma_h &= \frac{8\pi}{3}a^{-4}\int q^2 \mathrm{d}q \frac{q^2}{\epsilon} f_0 \varphi_2\,. \end{aligned} \tag{1.114}$$

We can then substitute the Legendre expansion in Eq. (1.91) and match the coefficients multiplying each Legendre polynomial (and the μ dependence on the metric perturbations). Following that approach and using the same recursion relation as above, we find that, in the Newtonian gauge,

$$\begin{aligned}
\varphi_0' &= -\frac{qk}{\epsilon}\varphi_1 + \Phi' \frac{\mathrm{d}\log f_0}{\mathrm{d}\log q}\,, \\
\varphi_1' &= \frac{qk}{3\epsilon}(\varphi_0 - 2\varphi_2) - \frac{\epsilon k}{3q}\Psi \frac{\mathrm{d}\log f_0}{\mathrm{d}\log q}\,, \\
\varphi_\ell' &= \frac{qk}{(2\ell+1)\epsilon}\left[\ell\varphi_{\ell-1} - (\ell+1)\varphi_{\ell+1}\right]\,, \qquad \ell \geq 2\,,
\end{aligned} \tag{1.115}$$

and, similarly, in synchronous gauge,

$$\begin{aligned}
\varphi_0' &= -\frac{qk}{\epsilon}\varphi_1 + \frac{1}{6}h' \frac{\mathrm{d}\log f_0}{\mathrm{d}\log q}\,, \\
\varphi_1' &= \frac{qk}{3\epsilon}(\varphi_0 - 2\varphi_2)\,, \\
\varphi_2' &= \frac{qk}{5\epsilon}(2\varphi_1 - 3\varphi_3) - \left(\frac{1}{15}h' + \frac{2}{5}\eta'\right) \frac{\mathrm{d}\log f_0}{\mathrm{d}\log q}\,, \\
\varphi_\ell' &= \frac{qk}{(2\ell+1)\epsilon}\left[\ell\varphi_{\ell-1} - (\ell+1)\varphi_{\ell+1}\right]\,, \qquad \ell \geq 3\,.
\end{aligned} \tag{1.116}$$

Note that in this case, the set of equations to solve is much larger, since due to the q dependence, we need to solve $\ell_{\max} \times N_{\mathrm{q}}$ equations, where $\ell_{\max}$ comes from the Boltzmann hierarchy and N_{q} comes from the number of evaluations in q used to approximate the q-integration for the phase-space distribution required to obtain the quantities that contribute to the stress-energy tensor, shown in Eq. (1.114).

1.4.4 Photons

Photons (which we will denote with γ) are massless particles that interact with baryons. Therefore, in this case we need to take into account the collision term in the Boltzmann equations, which describes the effects of the Compton scattering. At zero-th order the distribution function follows an unperturbed Bose-Einstein distribution. This is because the collision term includes the forward and backward reactions and we assume photons are in equilibrium, hence both reactions cancel and we can assume a Bose-Einstein distribution with no collision term. However, the perturbations from the unperturbed phase distributions are going to be determined by the collision term.

1.4.4.1 Derivation of the Compton collision term

Here we will derive the collision term. To do so we will have to rename some variables, that are not common to other sections in this course. The derivation is similar to the general discussion for the homogeneous Boltzmann equation. The scattering process of interest is

$$e^-(\boldsymbol{q}) + \gamma(\boldsymbol{p}) \longleftrightarrow e^-(\boldsymbol{q'}) + \gamma(\boldsymbol{p'})\,, \tag{1.117}$$

where the proper momentum of each particle is indicated between parentheses. We are interested in $f(\boldsymbol{p})$, so we need to integrate over the other three momenta. From Eq. (1.17), and denoting electron quantities with a subscript 'e',

$$
\begin{aligned}
C[f(\boldsymbol{p})] = \frac{1}{2E(p)} \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3 2E_e(q)} \int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3 2E(p')} \int \frac{\mathrm{d}^3\boldsymbol{q}'}{(2\pi)^3 2E_e(q')} \sum_{3\,\mathrm{spin}} |\mathcal{M}|^2 \times \\
\times (2\pi)^4 \delta_{\mathrm{D}}^{(3)}(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}')\delta_{\mathrm{D}}^{(1)}\left[E(p)+E_e(q)-E(p')-E_e(q')\right] \times \\
\times \{f(\boldsymbol{p}')f_e(\boldsymbol{q}')\,[1+f(\boldsymbol{p})]\,[1-f_e(\boldsymbol{q})] - \\
- f(\boldsymbol{p})f_e(\boldsymbol{q})\,[1+f(\boldsymbol{p}')]\,[1-f_e(\boldsymbol{q}')]\} \ .
\end{aligned}
\tag{1.118}
$$

We have explicitly included the sum over the final spin states of the outgoing electron and photon (two each) and the electron with which the incoming photon scatters. The Pauli blocking factors $1-f_e$ can be neglected since after electron-positron annihilation the occupation numbers are very small and this factor is never important. The photon energies are simply $E(p)=p$, and we assume the non-relativistic limit for the electrons, since kinetic energies are of the order of the temperature, much smaller than the electron mass at the times of interest.[11] Then, $E_e(q)-m_e = q^2/2m_e \sim T$, which means $q \sim T\sqrt{2m_e/T}$. Since $m_e/T \gg 1$, the electron momenta are much larger than the photon momenta.

We can perform the integral over $\boldsymbol{q}'$ easily using the three-dimensional Dirac delta:

$$
\begin{aligned}
C[f(\boldsymbol{p})] = \frac{\pi}{2pm_e} \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3 2m_e} \int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3 2E(p')} \times \\
\times \delta_{\mathrm{D}}^{(1)}\left[p+E_e(q)-p'-E_e(|\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}'|)\right] \sum_{3\mathrm{spin}} |\mathcal{M}|^2 \times \\
\times \{f(\boldsymbol{p}')f_e(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}')\,[1+f(\boldsymbol{p})] - f(\boldsymbol{p})f_e(\boldsymbol{q})\,[1+f(\boldsymbol{p}')]\} \ .
\end{aligned}
\tag{1.119}
$$

We have been talking about Compton scattering, but in reality we always work in the non-relativistic limit for electrons, hence the Thomson scattering. In this limit, the energy transferred is very small:

$$
p'-p = E_e(q) - E_e(|\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}'|) = \frac{q^2}{2m_e} - \frac{(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}')^2}{2m_e} \simeq \frac{(\boldsymbol{p}'-\boldsymbol{p})\boldsymbol{q}}{m_e} \ , \tag{1.120}
$$

where the last equality uses that $q \gg p,\, p'$. Since p and p' are of the same order, the right-hand side of the equation above is at most of order $2pq/m_e$. Then, the fractional change in photon energy is at most

$$
\frac{|p'-p|}{p} \lesssim \frac{2q}{m_e} \ll 1 \, , \tag{1.121}
$$

[11] This implies that except for small energy differences (as in the Dirac delta, we will always assume $E_e = m_e$).

which means that the non-relativistic Compton scattering is nearly elastic and $p' \simeq p$. Therefore, we can expand the Dirac delta function above as

$$\begin{aligned} &\delta_{\mathrm{D}}^{(1)}\left[p + E_e(q) - p' - E_e(|\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{p}'|)\right] \simeq \\ &\quad \simeq \delta_{\mathrm{D}}^{(1)}(p - p') + \frac{(\boldsymbol{p} - \boldsymbol{p}')\boldsymbol{q}}{m_e} \frac{\partial}{\partial p} \delta_{\mathrm{D}}^{(1)}(p - p') , \end{aligned} \tag{1.122}$$

where the partial derivative in the last term can equally be as function of p or p', changing the sign. Similarly, we will consider that $f_e(\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{p}') \simeq f_e(\boldsymbol{q})$. Thus, we have

$$\begin{aligned} C[f(\boldsymbol{p})] = &\frac{\pi}{8pm_e^2} \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} f_e(\boldsymbol{q}) \int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3 p'} \sum_{3\mathrm{spin}} |\mathcal{M}|^2 \times \\ &\times \left\{ \delta_{\mathrm{D}}^{(1)}(p - p') + \frac{(\boldsymbol{p} - \boldsymbol{p}')\boldsymbol{q}}{m_e} \frac{\partial}{\partial p'} \delta_{\mathrm{D}}^{(1)}(p - p') \right\} \times \\ &\times \left\{ f(\boldsymbol{p}') \left[1 + f(\boldsymbol{p})\right] - f(\boldsymbol{p}) \left[1 + f(\boldsymbol{p}')\right] \right\} , \end{aligned} \tag{1.123}$$

where we can see that the enhancing factors cancel each other. Now we can introduce the amplitude square for the Compton scattering. In the low-energy limit, and averaging over polarization states, this is

$$\frac{1}{2} \sum_{3\,\mathrm{spins}} |\mathcal{M}|^2 = 24\pi\sigma_{\mathrm{T}} m_e^2 \left(1 + \left[\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}'\right]^2\right) , \tag{1.124}$$

where σ_{T} is the Thomson cross-section. We will consider first the isotropic Compton scattering (averaging over angles) and then the anisotropic Compton scattering. Finally, there are also some effects from the fact that the temperature and polarization are coupled; we will not derive those terms, and will introduce them later.

First, averaging over angles, we have

$$\sum_{3\mathrm{spins}} |\mathcal{M}|^2 = 32\pi\sigma_{\mathrm{T}} m_e^2 , \qquad (\text{spin, pol., and angle averaged}) . \tag{1.125}$$

Now we can substitute this in Eq. (1.123) and compute the integrals, keeping only linear-order terms. The integral of f_e over $\boldsymbol{q}$ with no further factor results in $n_e/2$ (where the 1/2 factor comes from the two spin states of the electron, $g_e = 2$). In turn, terms that have a factor $\boldsymbol{q}/m_e$ yield $n_e\boldsymbol{v}_b$ factor after integration, where $\boldsymbol{v}_b$ is the bulk velocity of the baryons (which is the same as the electrons). Recovering the definition of the perturbed phase-space distribution $f = f_0(1+\varphi)$, we have

$$C[f(\boldsymbol{p})] = \frac{2\pi n_e\sigma_\mathrm{T}}{p}\int\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3p'}\left\{\delta_\mathrm{D}^{(1)}(p-p') + (\boldsymbol{p}-\boldsymbol{p}')\boldsymbol{v}_b\frac{\partial}{\partial p}\delta_\mathrm{D}^{(1)}(p-p')\right\}\times$$
$$\times\left\{f_0(p') - f_0(p) + f_0(p')\varphi(\boldsymbol{p}') - f_0(p)\varphi(\boldsymbol{p})\right\}\,,$$
$$= \frac{n_e\sigma_\mathrm{T}}{4\pi p}\int\mathrm{d}p'p'\int\mathrm{d}\Omega_{\mathrm{p}'}\left\{\delta_\mathrm{D}^{(1)}(p-p')\left[f_0(p')\varphi(\boldsymbol{p}') - f_0(p)\varphi(\boldsymbol{p})\right] + \right.$$
$$\left. +(\boldsymbol{p}-\boldsymbol{p}')\boldsymbol{v}_b\frac{\partial}{\partial p'}\delta_\mathrm{D}^{(1)}(p-p')\left[f_0(p') - f_0(p)\right]\right\}\,, \tag{1.126}$$

where we have already removed the background phase-space distributions in the first term, since they cancel due to the Dirac delta. We can use the monopole φ_0 of the distribution perturbation

$$\varphi_0 = \frac{1}{4\pi}\int\mathrm{d}\Omega_{\mathrm{p}'}\varphi\,, \tag{1.127}$$

which is the fractional perturbation in the angle-averaged photon flux. Since this monopole changes with position, it cannot be absorbed in f_0. Since $\boldsymbol{v}_b$ does not depend on the direction of the momentum, the second $\boldsymbol{p}'$ term in the integral vanishes. Therefore, after integrating over angles we have

$$C[f(\boldsymbol{p})] = \frac{n_e\sigma_\mathrm{T}}{p}\int\mathrm{d}p'p'\left\{\delta_\mathrm{D}^{(1)}(p-p')\left[f_0(p')\varphi_0 - f_0(p)\varphi(\boldsymbol{p})\right] + \right.$$
$$\left. +\boldsymbol{p}\boldsymbol{v}_b\frac{\partial}{\partial p'}\delta_\mathrm{D}^{(1)}(p-p')\left[f_0(p') - f_0(p)\right]\right\}\,. \tag{1.128}$$

Now we can integrate over p': the first integral is trivial, and the second can be done integrating by parts. In total we have:

$$C[f(\boldsymbol{p})] = n_e\sigma_\mathrm{T}f_0\left(\varphi_0 - \varphi - \frac{\mathrm{d}\log f_0}{\mathrm{d}\log p}\hat{\boldsymbol{p}}\boldsymbol{v}\right)\,. \tag{1.129}$$

Now we can transform this expression to the notation we have used in previous results. First, this collision term has been derived in terms of the proper time t, so we need to multiply by a to convert it to conformal time. Second, we can use $\mathcal{F}_\gamma$ in analogy to Eq. (1.106). And finally, we note that the $\left[\int p^2\mathrm{d}pf_0\mathrm{d}\log f_0/\mathrm{d}\log p\right]/\int p^2\mathrm{d}pf_0 = -4$ for bosons, and since $\hat{\boldsymbol{p}} = \hat{\boldsymbol{q}}$, we can express the isotropic, polarization-averaged collision term for the Boltzmann equation of $\mathcal{F}_\gamma$ as

$$C[\mathcal{F}_\gamma] = an_e\sigma_\mathrm{T}\left[-\mathcal{F}_\gamma + \mathcal{F}_{\gamma 0} + 4\hat{\boldsymbol{q}}\boldsymbol{v}_b\right]\,. \tag{1.130}$$

Now let us consider the anisotropic part. The squared amplitude for the anisotropic non-relativistic Compton scattering is $24\pi\sigma_\mathrm{T}m_e^2[(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}')^2-1/3]$, where term between square brackets is $(2/3)\mathcal{P}_2(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}')$. Expressing the Legendre polynomial in terms of the spherical harmonics $Y_{2m}(\hat{\boldsymbol{p}})$,

$$|\mathcal{M}|^2 = 16\pi\sigma_{\rm T} m_e^2 \frac{4\pi}{5} \sum_{m=-2}^{2} Y_{2m}(\hat{\boldsymbol{p}}) Y^*_{2m}(\hat{\boldsymbol{p}}') \,. \tag{1.131}$$

We use this squared amplitude in the collision term and only the $m = 0$ component survives, since the rest have an azimuthal dependence that integrates to zero. For a mode $\boldsymbol{k}$ this turns to be

$$\begin{aligned} C[f(\boldsymbol{p})] = \frac{n_e\sigma_{\rm T}}{8\pi p}\mathcal{P}_2(\mu) \int {\rm d}p' p' \int {\rm d}\Omega_{\rm p'} \mathcal{P}_2(\hat{\boldsymbol{k}}\hat{\boldsymbol{p}}') \times \\ \times \left\{ \delta_{\rm D}^{(1)}(p - p') + (\boldsymbol{p} - \boldsymbol{p}')\boldsymbol{v}_b \frac{\partial}{\partial p'} \delta_{\rm D}^{(1)}(p - p') \right\} \left[f(\boldsymbol{p}') - f(\boldsymbol{p}) \right] \,, \end{aligned} \tag{1.132}$$

where we have used that $Y_{20} = \sqrt{5}\mathcal{P}_2/\sqrt{4\pi}$. Computing the angular integral, the only term which survives at linear order is $\delta_{\rm D}^{(1)}(p-p')f(\boldsymbol{p}')$, which leaves

$$C[f(\boldsymbol{p})] = \frac{n_e\sigma_{\rm T}}{2p}\mathcal{P}_2(\mu) \int {\rm d}p' p' \delta_{\rm D}^{(1)}(p - p') f_0(p') \int \frac{{\rm d}\mu'}{2} \mathcal{P}_2(\mu')\varphi(\mu') \,, \tag{1.133}$$

where the integral over μ' returns $-\varphi_2$. Following the same procedure as above, this contributes with a $-\mathcal{F}_{\gamma 2}\mathcal{P}_2(\mu)/2$ to $C[\mathcal{F}_\gamma]$.

We have considered the momentum-averaged total phase-space density perturbation, summed over polarizations. But as mentioned above, polarization and intensity are coupled for photons, and we need to take into account the difference $\mathcal{G}_\gamma$ between the two linear polarization components. Accounting for this contribution, we have to consider the Boltzmann equation for both $\mathcal{F}_\gamma$ and $\mathcal{G}_\gamma$, which satisfy Eq. (1.91) with a right-hand side given by

$$\begin{aligned} C[\mathcal{F}_\gamma] = \left(\frac{\partial \mathcal{F}_\gamma}{\partial \tau}\right)_C &= a n_e \sigma_{\rm T} \left[-\mathcal{F}_\gamma + \mathcal{F}_{\gamma 0} + 4\hat{\boldsymbol{q}}\boldsymbol{v}_b - \frac{\mathcal{F}_{\gamma 2} + \mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}}{2}\mathcal{P}_2(\mu) \right] \,, \\ C[\mathcal{G}_\gamma] = \left(\frac{\partial \mathcal{G}_\gamma}{\partial \tau}\right)_C &= a n_e \sigma_{\rm T} \left[-\mathcal{G}_\gamma + \frac{\mathcal{F}_{\gamma 2} + \mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}}{2}(1 - \mathcal{P}_2(\mu)) \right] \,. \end{aligned} \tag{1.134}$$

Now we can proceed as for the case in the massless neutrinos: we expand $\mathcal{F}_\gamma$ and $\mathcal{G}_\gamma$ in Legendre series and use the relations $\hat{\boldsymbol{q}}\boldsymbol{v}_b = -(i\theta_b/k)\mathcal{P}_1(\mu)$ and those analog to Eq. (1.107), we rewrite the collision terms as

$$\begin{aligned} C[\mathcal{F}_\gamma] = a n_e \sigma_{\rm T} \Bigg[\frac{4i}{k}(\theta_\gamma - \theta_b)\mathcal{P}_1(\mu) + \left(9\sigma_\gamma - \frac{\mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}}{2} \right) \mathcal{P}_2(\mu) \\ - \sum_{\ell \geq 3} (-i)^\ell (2\ell + 1) \mathcal{F}_{\gamma \ell} \mathcal{P}_\ell(\mu) \Bigg] \,, \end{aligned} \tag{1.135}$$

and

$$C[\mathcal{G}_\gamma] = an_e\sigma_{\rm T}\left[\frac{1}{2}\left(\mathcal{F}_{\gamma 2} + \mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}\right)\left(1 - \mathcal{P}_2(\mu)\right) - \right. \\ \left. - \sum_{\ell \geq 0}(-i)^\ell(2\ell+1)\mathcal{G}_{\gamma\ell}\mathcal{P}_\ell(\mu)\right] . \tag{1.136}$$

1.4.4.2 Evolution equations

Once we have the collision term, we can proceed as we did for the massless neutrinos. Then, expanding the terms in the left-hand side of the Boltzmann equations in Legendre polynomials and matching the angular dependences, we find the evolution equations for Newtonian gauge

$$\begin{aligned}
\delta_\gamma' &= -\frac{4}{3}\theta_\gamma - 4\Phi' , \\
\theta_\gamma' &= k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\Psi + an_e\sigma_{\rm T}(\theta_b - \theta_\gamma) , \\
\mathcal{F}_{\gamma 2}' = 2\sigma_\gamma' &= \frac{8}{15}\theta_\gamma - \frac{3}{5}k\mathcal{F}_{\gamma 3} - \frac{9}{5}an_e\sigma_{\rm T}\sigma_\gamma + \frac{1}{10}an_e\sigma_{\rm T}(\mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}) , \\
\mathcal{F}_{\gamma\ell}' &= \frac{k}{2\ell+1}\left[\ell\mathcal{F}_{\gamma\ell-1} - (\ell+1)\mathcal{F}_{\gamma\ell+1}\right] - an_e\sigma_{\rm T}\mathcal{F}_{\gamma\ell} , \qquad \ell \geq 3 , \\
\mathcal{G}_{\gamma\ell}' &= \frac{k}{2\ell+1}\left[\ell\mathcal{G}_{\gamma\ell-1} - (\ell+1)\mathcal{G}_{\gamma\ell+1}\right] + \\
&\quad + an_e\sigma_{\rm T}\left[-\mathcal{G}_{\gamma\ell} + \frac{1}{2}\left(\mathcal{F}_{\gamma 2} + \mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}\right)\left(\delta_{\ell 0} + \frac{\delta_{\ell 2}}{5}\right)\right] ,
\end{aligned} \tag{1.137}$$

and, in synchronous gauge,

$$\begin{aligned}
\delta_\gamma' &= -\frac{4}{3}\theta_\gamma - \frac{2}{3}h' , \\
\theta_\gamma' &= k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + an_e\sigma_{\rm T}(\theta_b - \theta_\gamma) , \\
\mathcal{F}_{\gamma 2}' = 2\sigma_\gamma' &= \frac{8}{15}\theta_\gamma - \frac{3}{5}k\mathcal{F}_{\gamma 3} + \frac{4}{15}h' + \frac{8}{5}\eta' - \frac{9}{5}an_e\sigma_{\rm T}\sigma_\gamma + \frac{1}{10}an_e\sigma_{\rm T}(\mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}) , \\
\mathcal{F}_{\gamma\ell}' &= \frac{k}{2\ell+1}\left[\ell\mathcal{F}_{\gamma\ell-1} - (\ell+1)\mathcal{F}_{\gamma\ell+1}\right] - an_e\sigma_{\rm T}\mathcal{F}_{\gamma\ell} , \qquad \ell \geq 3 , \\
\mathcal{G}_{\gamma\ell}' &= \frac{k}{2\ell+1}\left[\ell\mathcal{G}_{\gamma\ell-1} - (\ell+1)\mathcal{G}_{\gamma\ell+1}\right] + \\
&\quad + an_e\sigma_{\rm T}\left[-\mathcal{G}_{\gamma\ell} + \frac{1}{2}\left(\mathcal{F}_{\gamma 2} + \mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}\right)\left(\delta_{\ell 0} + \frac{\delta_{\ell 2}}{5}\right)\right] .
\end{aligned} \tag{1.138}$$

Note that, as in the case for neutrinos, there is an infinite Boltzmann hierarchy that also needs to be closed, or solved using integral equations.

Let us take a short detour here. While we have preferred to decompose the phase-space distribution between the background unperturbed value (i.e., the Bose-Einstein distribution f_0 for photons) and a perturbation, we can also expand the Bose-Einstein distribution in terms of a temperature perturbation $\Theta \equiv (T - \bar{T})/\bar{T}$. Then, in this case we have

$$f = f_0 \left(\frac{q}{1+\Theta} \right) , \tag{1.139}$$

such as, by definition,

$$\Theta = - \left(\frac{\mathrm{d} \log f_0}{\mathrm{d} \log q} \right)^{-1} \varphi \,. \tag{1.140}$$

Since both the gravitational source terms and the linearized collision term in the Boltzmann equation for φ are proportional to the logarithmic derivative of f_0, Θ is independent of q. This means that the photon perturbations still have a Planck spectrum with a temperature that only depends on the photon direction (and not its moment).[12] From the equation above, we see that $\Theta = \mathcal{F}_\gamma/4$, which also relates the photon density and temperature perturbations by the same factor.

1.4.5 Baryons

The last component that we will study are the baryons. This misnomer is motivated by the fact that most of the energy density is dominated by the proton and neutron masses (since electrons are much lighter and heavier metals are much less abundant), and by the fact that Coulomb scattering (which couples protons and electrons) has a rate that is much larger than the expansion rate at all times of interest, which makes that the perturbations of all particles are the same. Hence, we will use the subscript 'b' for all of them collectively.

Baryons can be treated as cold and non relativistic, and therefore we will consider only the first two moments of their Boltzmann equations, as we did for dark matter. However, in this case we need to take also into account the coupling with photons due to the Compton scattering. Hence, the left hand side of the Boltzmann equations have the same form than for the dark matter. At the epochs of interest (around and after recombination), the reactions that change the number of electrons and nucleons (e.g., pair production, annihilation, etc.) are rare and therefore irrelevant. This means that there is no source term for the continuity equation, and thus the zero-th moment of the Boltzmann equation is as for cold dark matter,

$$\delta_b' = -\theta_b - 3\Phi' \,. \tag{1.141}$$

[12] This does not hold for nonlinear perturbations, which indeed change the spectrum of the CMB. This is the case of for instance the Sunyaev-Zeldovic effect, among many other processes, that generate what is known as spectral distortions: deviations from the black-body spectrum of the CMB.

While the number of baryons is conserved, their momentum is not, since there is momentum transfer with the photons. The derivation of the second moment is similar than for the dark matter, but instead of weighting the integrals by $\boldsymbol{p}/E$, we use only $\boldsymbol{p}$, which makes the cold dark matter derivation correct if we multiply by a factor of mass m. Since the proton mass vastly dominates, we have

$$m_p \frac{\partial(n_b v_b^j)}{\partial\tau} + 4\mathcal{H} m_p n_b v_b^j + i m_p n_b k^j \Psi = F_{e\gamma}^j\,, \tag{1.142}$$

where m_p is the proton mass and the force density $\boldsymbol{F}_{e\gamma}$ encodes the momentum transfer between photons and electrons due to Compton scattering.[13] Dividing both sides by $\bar{\rho}_b = m_p \bar{n}_b$ we are left with

$$\frac{\partial v_b^j}{\partial\tau} + \mathcal{H} v_b^j + i k^j \Psi = \frac{F_{e\gamma}^j}{\bar{\rho}_b}\,. \tag{1.143}$$

We have left to compute the momentum transfer between photons and electrons. Since momentum is conserved, the force term has to be precisely equal and opposite to the force term in the photon analog of the baryon Euler equation. Therefore, momentum conservation introduces a term $(4\bar{\rho}_\gamma/3\bar{\rho}_b) a n_e \sigma_{\mathrm{T}}(\theta_\gamma - \theta_b)$, where the prefactors in the mean densities come from the different time dependence for each component.

In addition there is another term coming from the baryon sound speed $c_s^2 = \delta P_b/\delta\rho_b$. This is because baryons, although being non relativistic, are not completely cold as dark matter (which we assume it has zero temperature). The finite temperature of baryons introduces this non-zero (although very small) sound speed, which can be neglected in all terms except the acoustic term $c_s^2 k^2 \delta$. The sound speed for baryons depends on the gas temperature, the evolution of which can also be tracked using the first law of thermodynamics. The perturbations of the gas temperatures are therefore coupled to the baryon perturbations and therefore to the whole system to solve, although its effect is limited to very small scales and usually neglected in most studies that involve only linear scales and do not depend directly in the gas temperature.

Then, in Newtonian gauge, we have

$$\begin{aligned} \delta_b' &= -\theta_b - 3\Phi'\,, \\ \theta_b' &= -\mathcal{H}\theta_b + k^2\Psi + c_s^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_{\mathrm{T}}(\theta_\gamma - \theta_b)\,, \end{aligned} \tag{1.144}$$

and in synchronous gauge,

$$\begin{aligned} \delta_b' &= -\theta_b - \frac{1}{2}h'\,, \\ \theta_b' &= -\mathcal{H}\theta_b + c_s^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_{\mathrm{T}}(\theta_\gamma - \theta_b)\,. \end{aligned} \tag{1.145}$$

[13] Electrons transfer the momentum to the nuclei immediately, and the nuclei-photon interaction is suppressed by a m_e^2/m_p^2 factor, hence neglected.

1.4.6 Others

We have considered the standard components of the Universe in the ΛCDM model, but this does not mean that there may be other components and new physics. New components, or new interaction between the standard species can be included in the system of differential equations that describe the evolution of the matter, radiation and metric perturbations in the Universe, following a similar analysis that we have done in this section.

CHAPTER 2

INFLATION

In the previous chapter we have discussed how perturbations evolve in an inhomogeneous expanding Universe, but we did not discuss how those perturbations arise or what are the actual initial conditions for them. The quest for understanding the initial conditions leads to the research about the primordial Universe. Besides aiming to describe the initial conditions of the Universe, studying the primordial Universe has led to the development of theories that solve some shortcomings of the Big Bang. The most studied and plausible theory to explain the primordial Universe and the creation of the initial perturbations is inflation, which proposes a primordial phase of exponential expansion of the Universe. Other alternatives include for instance the ekpyrosis or bouncing Universes, which proposes a slowly contracting phase of the Universe that *bounces* in a transition to the presently observed expanding Universe. This *bounce* turns out to be difficult to control in the computations. We will focus on inflation, having in mind that many of the tools and arguments discussed also applies to theories like ekpyrosis.

We will first describe the main flaws of the Big Bang to explain some of the observations and measured properties of the Universe, discuss the basic dynamics of inflation to solve them and how they lead to the primordial perturbations that act as initial conditions for the evolution of the perturbations studied in the previous chapter.

2.1 Big Bang problems

The Big Bang explains satisfactorily the thermal history of the Universe, as well as many properties that we can observe and measure today. However, it has some shortcomings regarding some measurements that cannot explain. This motivated the study of the primordial Universe and the developments of theories like inflation. Some of these shortcomings are the horizon and the flatness problem.

The Universe was, in its early states, extremely homogeneous. In particular, measurements of the CMB confirm that, at the time of recombination (roughly when the Universe was 380 000 years old, or at a redshift of 1100), the size of the inhomogeneities was roughly 1 part in 10^5, with its components being very close to thermal equilibrium. Given enough time, any fluid will reach equilibrium in the absence of external forces, even if the initial state was very homogeneous. However, this is true for a causally connected volume. The impact of forces, etc., is not instant: the mediator particles have to travel propagating the force. Therefore, we can consider two points to be causally connected if there has been enough time to travel between them. This is not the case for our universe, since different parts of the Universe probed by CMB observations were not in causal contact.

We can quantify this statement with the comoving horizon, which is the comoving distance $\int_0^t \mathrm{d}t'/a(t)$ that light can travel between two times.[1] The comoving horizon coincides with the conformal time τ (remember $\mathrm{d}t = a\mathrm{d}\tau$).[2] This is why it is common to express the wavenumber k_h associated to the scale of the horizon as the one fulfilling $k_\mathrm{h}\tau \sim k_\mathrm{h}/aH = k_\mathrm{h}/\mathcal{H} \sim 1$.

Two patches in the sky separated by a small angle α are separated, at the time of the CMB, a comoving distance

$$\chi(\alpha) \simeq \chi_* \alpha = (\tau_0 - \tau_*)\alpha \,, \tag{2.1}$$

where τ_* and τ_0 are the conformal times at recombination and today. In the concordance cosmological model, $\tau_* \approx 280h^{-1}$ Mpc, and $\tau_0 \approx 14200h^{-1}$ Mpc. Therefore, two patches separated by $\alpha \gtrsim \tau_*/(\tau_0 - \tau_*) \approx 1.2^\circ$ are causally disconnected and therefore there is no reason for them to be in thermal equilibrium. This is known as the *horizon* problem.

[1] Note that there is a c factor in the integral that is set to 1 by the use of natural units.
[2] For this reasons, in some numerical computations, such as those undertaken in Boltzmann codes, the conformal time is measured in Mpc.

If we express the comoving horizon in terms of the scale factor,

$$\tau(a) = \int_0^a \mathrm{d}\log a' \frac{1}{\mathcal{H}}\,, \tag{2.2}$$

the comoving horizon is the logarithmic integral of the conformal Hubble radius, which is the approximate distance over which light can travel during one expansion time (i.e., the time that takes the scale factor to increase by a factor of e). For a radiation or matter dominated Universe, $\mathcal{H}^{-1} \propto a$ and $a^{1/2}$, respectively, such as the conformal Hubble radius always grows and the largest contributions to the comoving horizon come from the most recent epochs (i.e., $a \sim 1$). For τ_* to be large enough to avoid the horizon problem, it would need a large contribution from very early times, and for that the primordial conformal Hubble radius would have to be much larger than now (or at least that at the time of recombination). This would allow for two volumes separated by a long distance to be causally-connected, solving the horizon problem. If we need $\mathcal{H}^{-1} = \dot{a}^{-1}$ to decrease, this means $\ddot{a}$ increasing, which asks for a primordial epoch of early acceleration. This postulated epoch is what we call inflation.

Another way to interpret how the horizon problem is solved is in term of the physical distance $\chi_{\rm phys} = \chi/a$ between two particles. At very early times, before the accelerated expansion, two particles that are very close, well within the comoving horizon, will be very far from each other (beyond the comoving horizon) after such accelerated inflation. This accelerated expansion would *effectively* empty out the Universe, since the number density of particles significantly decreases. This last argument would serve as another explanation for why the Universe is so smooth, but would mean that the Universe is empty. This is fortunately not the case, since at the end of inflation, the particles driving inflation are converted to ordinary particles which quickly thermalize. This process is referred to as reheating.

Then, inflation produces that scales that were causally connected become superhorizon during the accelerated expansion to, after the Universe expands more slowly during the radiation and matter dominated eras, enter again the horizon. We can quantify for how long inflation should last. Let us assume that the temperature of the Universe just after inflation ends is $T_{\rm e}$. Assuming radiation domination, $\mathcal{H} \propto a^{-1}$ the ratio between the conformal Hubble radius at that moment and today is $\mathcal{H}_0/\mathcal{H}_{\rm e} = a_{\rm e}/a_0$. Since $T \propto a^{-1}$, $a_{\rm e}/a_0 \sim T_0/T_{\rm e} \sim 10^{-13}/T_{\rm e}$ GeV. Assuming that the energy scale of the end of inflation is $\sim 10^{14}$ GeV, this would mean that the conformal Hubble radius at the end of inflation was 27 orders of magnitude smaller than it is today. We conclude that the scale factor had to increase by a factor of $10^{27} \sim e^{62}$ during inflation. Assuming a constant Hubble rate, the expansion is exponential, and we can say that the Universe had to expand exponentially for ~ 62 e-folds.

The accelerated expansion naturally solves another problem of the Big Bang, the *flatness problem.* If matter curves the Universe, why is the Universe so close to a flat Euclidean space? This would involve an extreme fine tuning of the curvature parameter κ of the Universe to zero. Defining $\Omega = \rho/\rho_{\rm crit}$, $-\kappa/\mathcal{H}^2 = 1-\Omega$. Since κ is constant and $\mathcal{H}^{-1}$ grows monotonically in the standard Big Bang cosmology, $|1-\Omega|$ grows with time, making $\Omega = 1$ an unstable point, forcing an initial value of Ω extremely close to unity. However, since inflation involves decreasing $\mathcal{H}^{-1}$ it significantly relaxes this assumption and actually turns $\Omega = 1$ an attractor solution. In other terms, after an exponential expansion, any non-zero curvature of the Universe would still appear as *effectively* flat.

Note that the comoving horizon (or conformal time) is not a meaningful measure of time, since it grows a lot at primordial times, and then it grows very slowly. Since it increases monotonically at all times, we are free to set its zero-point. Therefore, we will consider that $\tau = 0$ corresponds to the moment of the end of inflation, and define $\tau = \int_{t_{\rm e}}^{t} {\rm d}t'/a(t')$.

2.2 Accelerated expansion

We have seen before, in the context of the discussion of the dark energy and the cosmological constant, that in order to have accelerated expansion we need an effective pressure below zero. As we discussed, we do not know any kind of matter that fulfills this requirement, which has led to theories attempting to explain at the same time inflation and dark energy. Anyways, we need a new, additional species to drive the accelerated expansion, and it cannot be a cosmological constant, because we do need inflation to eventually finish.

The simplest possibility is the potential energy of a scalar field.[3] Although there are other possibilities involving more degrees of freedom, we will restrain our discussion to this model. We will denote this scalar field by ϕ,[4] which has an energy momentum tensor

$$T^{\alpha}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{\beta}} - \delta^{\alpha}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right] , \tag{2.3}$$

where $V(\phi)$ is the potential for the field. For example, a free field with mass m has a potential $V = m^2\phi^2/2$. We can also treat ϕ perturbatively, and consider it homogeneous at background level. Therefore, focusing on background quantities, only time derivatives are important, and we have

$$T^{\alpha}_{\beta} = -\delta^{\alpha}_{0}\delta^{0}_{\beta}\dot{\phi}^2 + \delta^{\alpha}_{\beta}\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right] , \tag{2.4}$$

[3]Indeed, one of the simplest models of dark energy beyond a cosmological constant is quintessence, which is also based on the inclusion of scalar field(s). Note, however, that both quintessence and inflation cannot be trivially explained by the only scalar field we know, the Higgs boson, since its properties are too constrained by now for us to know that we cannot make it work for these purposes.

[4]Not to be confused with the metric perturbation in the Newtonian gauge Φ.

Note that for a homogeneous scalar field the stress-energy tensor is the diagonal $\{-\rho, P, P, P\}$. For the time-time component $T^0_0 = -\rho$, so that

$$\rho = \frac{\dot\phi^2}{2} + V(\phi)\,, \tag{2.5}$$

which are the kinetic and potential energy densities of the field: a homogeneous scalar field has the same dynamics as a single particle in a potential. The pressure $P = T^i_i$ is

$$P = \frac{\dot\phi^2}{2} - V(\phi)\,. \tag{2.6}$$

Therefore, if the potential energy is larger than the kinetic energy, a negative pressure is possible. We can see the same in terms of the equation of state

$$w = \frac{P}{\rho} = \frac{\dot\phi^2/2 - V(\phi)}{\dot\phi^2/2 + V(\phi)}\,, \tag{2.7}$$

for which we approximate the behaviour (at background level) of a cosmological constant, e.g., $w = -1$, if $V(\phi) \gg \dot\phi^2$. This can be achieved with a very flat potential, for which the scalar field changes very slowly, which is known by slow-roll inflation. In this scenario inflation ends once the scalar fields gets over the flat part of the potential and reaches its minimum, where it oscillates and decay into lighter particles. Many different forms of potential have been (and are!) proposed, especially once additional degrees of freedom are allowed (multifield inflation, etc). The observable consequences of inflation are only a few, and many different models can be designed to fit current obsservations. This is why we will not specify any model and will treat V in general.

The evolution of ϕ can be obtained from the conservation of the energy momentum tensor, which for the homogeneous case returns

$$\frac{\partial\rho}{\partial t} + 3H(\rho + P) = 0\,. \tag{2.8}$$

If we substitute the expressions above for the density and pressure and divide by $\dot\phi$ we have

$$\ddot\phi + 3H\dot\phi + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,, \tag{2.9}$$

which is Klein-Gordon equation. In conformal time, we have

$$\phi'' + 2\mathcal{H}\phi' + a^2\frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,. \tag{2.10}$$

In slow-roll models, the zero-th order field, and hence the Hubble rate, vary very slowly. Assuming a constant Hubble rate, we find

$$\tau = \int_{a_\mathrm{e}}^{a} \frac{\mathrm{d}a}{Ha^2} \simeq \frac{1}{H}\int_{a_\mathrm{e}}^{a}\frac{\mathrm{d}a}{a^2} \simeq -\frac{1}{Ha}\,, \tag{2.11}$$

where the second equality uses that $a_e \gg a$, that the scale factor at the end of inflation is much larger than during inflation, due to the exponential expansion.

To quantify slow-roll, we can define two variables that vanish in the limit in which ϕ is perfectly constant. There are many conventions, but we will use one of the most directly linked to observables. First,

$$\epsilon_{\rm sr} \equiv \frac{\rm d}{{\rm d}t} H^{-1} = -\frac{\dot{H}}{H^2} = -\frac{H'}{aH^2}\,, \tag{2.12}$$

yields the fractional change during an e-fold in the Hubble rate. Since H decreases, $\epsilon_{\rm sr}$ is always positive: during radiation domination, $\epsilon_{\rm sr} = 2$, but during inflation, $\epsilon_{\rm sr}$ will be very small. Note that an alternative definition is

$$\epsilon_{\rm sr} - 1 = \frac{\rm d}{{\rm d}\tau}(aH)^{-1}\,. \tag{2.13}$$

The second variable quantifies how slowly the field rolls:

$$\delta_{\rm sr} \equiv \frac{1}{H}\frac{\ddot{\phi}}{\dot{\phi}} = -\frac{aH\phi' - \phi''}{aH\phi'} = -\frac{3aH\phi' + a^2\frac{{\rm d}V}{{\rm d}\phi}}{aH\phi'}\,, \tag{2.14}$$

where the last equality uses the Klein-Gordon equation derived above. These two parameters quantify key features of the inflationary predictions, regarding the primordial curvature perturbations and the production of gravitational waves.

2.3 Primordial scalar perturbations

Inflation, by design and to solve the horizon problem, correlates scales that would otherwise be causally disconnected and therefore uncorrelated. The discussion above ensures that the Universe is close to homogeneous, but there are still some minuscule perturbations around the background values. These perturbations are generated primordially when the scales are causally connected and survive after inflation. In this section we will discuss these perturbations, focusing especially on those that are scalar, since are the ones coupled to density and responsible for the structures that we observe in the Universe. In addition to scalar perturbations, inflation also generates tensor perturbations, which propagate in the form of gravitational waves. These perturbations are not coupled to the density and therefore are not responsible for the large-scale structure of the Universe, but they do induce anisotropies in the CMB, in particular in the B modes of polarization.[5]

[5] This is probably the most promising way that we have to directly measure the consequences of inflation today, since scalar perturbations do not generate B-mode polarization power spectrum. However, the signal is very small, obscured by foreground contamination, and also by the conversion from E-mode to B-modes due to secondary anisotropies such as CMB lensing.

2.3.1 Kinds of scalar perturbations

At any given point and time during inflation, there are small perturbations due to quantum fluctuations of the field against the uniform background. Statistically, the mean fluctuation is null because overdensity regions cancel with underdensities. However, the variance of these perturbations is not zero, and will be the main focus of our study. In principle we would have to specify the predicted perturbations for each species that results from inflation. In general, we can distinguish between two kind of perturbations: adiabatic and isocurvature perturbations.

Adiabatic perturbations fulfill that the local state of matter at some space time point of the perturbed Universe is the same as in the background at some slightly different time (where the time shift varies with the location). One way to understand adiabatic perturbations is to interpret that some regions of the Universe are *ahead* or *more evolved* than others. This local shift in time is *common* to all species involved, fulfilling that

$$\delta\rho(\tau,\boldsymbol{x}) = \bar{\rho}(\tau+\delta\tau(\boldsymbol{x}),\boldsymbol{x}) = \bar{\rho}'(\tau)\delta\tau(\boldsymbol{x}) \tag{2.15}$$

for all species, which means that

$$\frac{\delta\rho_x}{\bar{\rho}'_x} = \frac{\delta\rho_y}{\bar{\rho}'_y}\,. \tag{2.16}$$

Neglecting any energy transfer between fluid components at the background level, $\bar{\rho}'_x = -3\mathcal{H}(1+w_x)\bar{\rho}_x$, so that

$$\frac{\delta_x}{1+w_x} = \frac{\delta_y}{1+w_y}\,. \tag{2.17}$$

Thus, all matter species have the same fractional perturbations, while all radiation and relativistic species obey $\delta_\gamma = 4\delta_m/3$, since $w_\gamma = 1/3$. The relation for the velocity divergences is analog.

On the other hand, instead of corresponding to a change in the total energy density, isocurvature perturbations correspond to perturbations between different species that explicitly leave the total perturbations unchanged. Therefore, isocurvature perturbations can be defined as

$$S_{xy} = \frac{\delta_x}{1+w_x} - \frac{\delta_y}{1+w_y}\,. \tag{2.18}$$

There are different sets of isocurvature perturbations, usually defined with respect to the photon perturbations (e.g., neutrino isocurvature perturbations involve neutrino and photons in the expression above).

Single-field inflation, since it involves a single clock (scalar field), predicts only the generation of adiabatic perturbations. This is because any point during inflation is completely characterized by the value of the single scalar field involved. Actually, isocurvature perturbations are very constrained by current observations of the CMB anisotropies. Some exceptions are compensated dark-matter-baryon isocurvature perturbations (i.e., isocurvature perturbations involving *only* dark matter and baryons). Anyways, since we are focusing on single-field inflation, we restrict the discussion to adiabatic perturbations and we only need to derive $\delta\rho$. We can therefore specify the initial conditions in terms of a single metric perturbations.

2.3.2 Scalar field perturbations

Let us decompose the scalar field in its background value and perturbations,

$$\phi(\boldsymbol{x}, t) = \bar{\phi}(t) + \delta\phi(\boldsymbol{x}, t)\,. \tag{2.19}$$

For a scalar field, the stress-energy tensor is given by Eq. (2.3). As it will be evident later, the impact of metric perturbations in the stress-energy tensor is negligible. Therefore, let us ignore so far (i.e., what is called as 'ignoring gravity'), for which the perturbations to the stress-energy tensor. In this limit, the metric is diagonal and $\{-1, a^2\}$. The time-space components of the metric are zero, so that $T^i_0 = g^{i\nu}(\mathrm{d}\phi/\mathrm{d}x^\nu)(\mathrm{d}\phi/\mathrm{d}t)$, and since the metric is diagonal and the background value of field is homogeneous,

$$\delta T^i_0 = \frac{ik^i}{a^3}\bar{\phi}'\delta\phi \tag{2.20}$$

at linear order. The time-time component can be derived in a similar way. If we expand $V(\bar{\phi} + \delta\phi) = V(\bar{\phi}) + \delta\phi\mathrm{d}V(\bar{\phi})/\mathrm{d}\phi$, the first-order perturbation is

$$\delta T^0_0 = -\frac{1}{a^2}\bar{\phi}'\delta\phi' - \delta\phi\frac{\mathrm{d}V(\bar{\phi})}{\mathrm{d}\phi}\,. \tag{2.21}$$

Finally, the space space component is

$$\delta T^i_j = \delta^i_j\left(\frac{1}{a^2}\bar{\phi}'\delta\phi' - \delta\phi\frac{\mathrm{d}V(\bar{\phi})}{\mathrm{d}\phi}\right)\,. \tag{2.22}$$

Note that the space-space component is diagonal, which means that there is no anisotropic stress, hence $\Psi = -\Phi$ during inflation. This reduces nicely the number of variables.

We can now then derive the evolution equations for $\delta\phi$ from the conservation of the stress-energy tensor,

$$\frac{\partial T^\mu_\nu}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu}T^\alpha_\nu - \Gamma^\alpha_{\nu\mu}T^\mu_\alpha = 0\,, \tag{2.23}$$

which accounting for the perturbations in ϕ and in the metric (assuming the Newtonian gauge), we have for the time component

$$\frac{\partial\delta T^0_0}{\partial t} + ik_i\delta T^i_0 + 3H\delta T^0_0 - H\delta T^i_i + 3(\bar{\rho} + \bar{P})\dot{\Psi} = 0\,, \tag{2.24}$$

where we have replaced $\dot{\Phi}$ by $-\dot{\Psi}$. During inflation we can neglect the last term: the Einstein equations yield $\Psi \sim \delta T^0_0/\bar{\rho}$, which makes all terms above but the last to be of the order of $\sim \bar{\rho}\Psi$, and remember that one of the conditions of slow roll is $|\bar{\rho} + \bar{P}| \ll \bar{\rho}$ (since $w \sim -1$). In Newtonian gauge, the connection between Ψ and $\delta\phi$ through the Einstein equation will build up a relationship between the two as inflation progresses. This relation does not affect our calculation of the evolution of $\delta\phi$, but will be critical to connect the perturbations in inflation to the metric, matter and radiation perturbations after inflation.

Substituting the values of the stress energy in the conservation of the perturbation of the stress-energy momentum of Eq. (2.24), we have

$$\left(\frac{1}{a}\frac{\partial}{\partial\tau} + 3H\right)\left(\frac{-\bar{\phi}'\delta\phi'}{a^2} - \delta\phi\frac{\mathrm{d}V}{\mathrm{d}\phi}\right) - \\ - \frac{k^2}{a^3}\bar{\phi}'\delta\phi - 3H\left(\frac{\bar{\phi}'\delta\phi'}{a^2} - \delta\phi\frac{\mathrm{d}V}{\mathrm{d}\phi}\right) = 0 \tag{2.25}$$

propagating the derivatives (note that $\partial/\partial\tau(\mathrm{d}V/\mathrm{d}\phi) = \bar{\phi}'\mathrm{d}^2V/\mathrm{d}\phi^2$) and multiplying by a^3, we have

$$-\bar{\phi}'\delta\phi'' + \delta\phi'\left(-\bar{\phi}'' - 4aH\bar{\phi}' - a^2\frac{\mathrm{d}V}{\mathrm{d}\phi}\right) + \delta\phi\left(-a^2\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\bar{\phi}' - k^2\bar{\phi}'\right) = 0\,. \tag{2.26}$$

The double derivative of V is typically small (proportional to the slow-roll parameters that are supposed to be very close to zero), so it can be neglected. The first parenthesis can be substituted using the Klein-Gordon equation for the background evolution of the field (it is equal to $-2aH\bar{\phi}'$), so after dividing by $\bar{\phi}'$ we have

$$\delta\phi'' + 2aH\delta\phi' + k^2\delta\phi = 0\,. \tag{2.27}$$

Now we can solve this equation in order to find the evolution of the perturbations of the field. Let us define $f \equiv a\delta\phi$, so that $\delta\phi' = f'/a - a'f/a^2$ and $\delta\phi'' = f''/a - 2a'f'/a^2 - a''f/a^2 + 2(a')^2 f/a^3$, which turns the differential equation into

$$f'' + \left(k^2 - \frac{a''}{a}\right)f = 0\,. \tag{2.28}$$

ϕ is a quantum field, and we will not discuss its quantum effects in detail. We can quantize f and express it in terms of the quantum operator

$$\hat{f}(\boldsymbol{k},\tau) = v(k,\tau)\hat{a}_{\boldsymbol{k}} + v^*(k,\tau)\hat{a}^\dagger_{\boldsymbol{k}}\,, \tag{2.29}$$

where $\hat{a}$ is the annihilation operator and v is the positive-frequency solution to the harmonic oscillator equation above. v then fulfills the same differential equation as f above, and can be used to estimate the variance of the field:

$$\langle \hat{f}^\dagger(\boldsymbol{k},\tau)\hat{f}(\boldsymbol{k}',\tau)\rangle = |v(\boldsymbol{k},\tau)|^2(2\pi)^3\delta^{(3)}_{\mathrm{D}}(\boldsymbol{k} - \boldsymbol{k}')\,. \tag{2.30}$$

Let us solve the differential equation then during inflation. Since $a' = a^2 \simeq -a/\tau$ (using Eq. (2.11)), $a''/a \simeq 2/\tau^2$. For $k|\tau| \gg 1$ the mode is well within the horizon, such as the k^2 term dominates and we have a simple harmonic oscillator with solution

$$v = \frac{e^{-ik\tau}}{\sqrt{2k}} \left[1 - \frac{i}{k\tau} \right] , \tag{2.31}$$

but as inflation has undergone enough e-folds the mode exits the horizon and $k|\tau|$ becomes very small, which leads to the limit

$$v = \frac{e^{-ik\tau}}{\sqrt{2k}} \frac{-i}{k\tau} . \tag{2.32}$$

If we now take the power spectrum of $\delta\phi = f/a$ (which scales as $|f^2|/a^2$, we find that the variance of the perturbations of the scalar field driving inflation is

$$P_{\delta\phi} = \frac{1}{2k^3 a^2 \tau^2} = \left(\frac{H^2}{2k^3} \right)_{\text{hor. cross.}} . \tag{2.33}$$

2.3.3 Curvature perturbations

The derivation in the previous subsection holds for scales that are way within the horizon, but the metric perturbation gets relevant by the end of inflation: as inflation progresses, a connection between $\delta\phi$ and Ψ arises, and this connection freezes (is conserved) outside of the horizon, and is therefore determined by the perturbation of the scalar field at horizon crossing. This will allow us to express the power spectrum P_Ψ of the primordial metric perturbations just after inflation in terms of the power spectrum $P_{\delta\phi}$ of the scalar field perturbation at horizon crossing.

Let us define the curvature perturbation

$$\mathcal{R}(\boldsymbol{k}, \tau) \equiv \frac{ik_i \delta T^i_0(\boldsymbol{k}, \tau) a^2 H(\tau)}{k^2 \left[\bar{\rho} + \bar{P} \right] (\tau)} - \Psi(\boldsymbol{k}, \tau) . \tag{2.34}$$

During inflation, the first term dominates, and we can express $\bar{\rho} + \bar{P} = (\bar{\phi}'/a)^2$ from the Klein-Gordon equation, and using the value of the perturbation of the stress energy tensor we have

$$\mathcal{R} = -\frac{aH}{\bar{\phi}'} \delta\phi , \qquad (\text{during inflation}) . \tag{2.35}$$

Enough time after inflation ends, when we are fully in the radiation dominated epoch, the stress-energy tensor is fully dominated by radiation, and we can assume $ik_i \delta T^i_0 = -k \bar{\rho}_r \mathcal{F}_{r1}$ (i.e., proportional to the dipole of the momentum-averaged phase-space distribution function perturbations from previous chapter). Note that at these times, neutrinos are completely relativistic, hence

$\mathcal{F}_r = \mathcal{F}_\gamma + \mathcal{F}_\nu$. Using the equation of state of radiation in the denominator of $\mathcal{R}$, we have

$$\mathcal{R} = -\frac{3aH\mathcal{F}_{r1}}{4k} - \Psi = -\frac{3}{2}\Psi\,, \qquad (\text{post inflation}\,;\text{rad. dom})\,. \tag{2.36}$$

The last equality will be derived in the next section, please be patient.

In order to relate the two values of $\mathcal{R}$ at these two times, we need to prove that it is actually conserved outside of the horizon. From the conservation of the stress-energy tensor perturbations in Eq. (2.24), if we take that large scale limit, since $k_i\delta T^i_0 \propto k^2$ (and $k \ll 1$ in this limit), we can neglect that term. Consider the perturbed Einstein equation dependent on the matter density (note that the right-hand side in Eq. (1.65) is $-4\pi Ga^2\delta T^0_0$), and the space-time component given by

$$ik_i\left(\Phi' - \mathcal{H}\Psi\right) = 4\pi Ga\delta T^0_i\,. \tag{2.37}$$

Accounting for fact that there is no anisotropic stress (i.e., $\Phi = -\Psi$), we can add them to obtain

$$k^2\Psi = 4\pi Ga^2\left[\delta T^0_0 + \frac{3a\mathcal{H}ik_i\delta T^i_0}{k^2}\right]\,. \tag{2.38}$$

In the large-scale limit we are considering, the left-hand side vanishes, and we have

$$\frac{a\mathcal{H}ik_i\delta T^i_0}{k^2} = -3\delta T^0_0\,. \tag{2.39}$$

If we go back to Eq. (2.34), we find that in the large-scale limit,

$$\mathcal{R} = -\Psi - \frac{1}{3}\frac{\delta T^0_0}{\bar\rho + \bar P}\,, \tag{2.40}$$

so that if we change Ψ by $\mathcal{R}$ in the conservation of the stress tensor, we have

$$\delta T^0_0\left[3H + \frac{\dot{\bar\rho} + \dot{\bar P}}{\bar\rho + \bar P}\right] - H\delta T^i_i = 3(\bar\rho + \bar P)\frac{\partial\mathcal{R}}{\partial t}\,. \tag{2.41}$$

Remembering that $\dot{\bar\rho} = -3H(\bar\rho + \bar P)$, the left hand side can be rewritten so that

$$3H\left[\frac{\dot{\bar P}}{\dot{\bar\rho}}\delta\rho - \delta P\right] = 3(\bar\rho + \bar P)\frac{\partial\mathcal{R}}{\partial t}\,. \tag{2.42}$$

During inflation, since we have a single field $\bar\phi$ that can be understood as a clock and therefore a time coordinate, we can express any background quantity $X = (\mathrm{d}X/\mathrm{d}\bar\phi)\dot{\bar\phi}$. This argument can also be used for the perturbations, using $\delta\phi$ instead. Therefore, in single field inflation

$$\delta P = \frac{\dot{\bar P}}{\dot{\bar\rho}}\delta\rho\,, \tag{2.43}$$

so that $\mathcal{R}$ is conserved in the large-scale limit. In more complicated inflationary models, this does not hold and the curvature perturbation evolves outside of the horizon.

Now that we have proven that the curvature perturbation is constant during inflation, we can relate Ψ after inflation with $\delta\phi$ at horizon crossing:

$$(\Psi)_{\text{post}} = \frac{2}{3} aH \left(\frac{\delta\phi}{\bar{\phi}'} \right)_{\text{hor. cross}} . \tag{2.44}$$

Therefore, the power spectrum of Ψ post inflation is

$$(P_\Psi)_{\text{post}}(k) = \frac{4}{9} \left(\frac{aH}{\bar{\phi}'} \right)^2 (P_{\delta\phi})_{aH=k} = \frac{2}{9k^3} \left(\frac{aH}{\bar{\phi}'} \right)^2_{aH=k} , \tag{2.45}$$

where we have used the value for the power spectrum of $\delta\phi$ from Eq. (2.33). Combining the Friedmann equations with the relations between $\dot{\bar{\phi}}$ and the density and pressure, find that $(aH/\bar{\phi}')^2 = 4\pi G/\epsilon_{\text{sr}}$, thus

$$(P_\Psi)_{\text{post}}(k) = (P_\Phi)_{\text{post}}(k) = \frac{8\pi G}{9k^3} \left(\frac{H^2}{\epsilon_{\text{sr}}} \right)^2_{aH=k} , \tag{2.46}$$

where we also use the non-anisotropic stress quality of $\Phi = -\Psi$. If a similar analysis is carried out for tensor perturbations (i.e., gravitational waves), we would find that the power spectrum of the amplitude of such perturbations obeys a tensor-to-scalar power spectra ratio of ϵ_{sr}, hence scalar perturbations are much larger than tensor perturbations. Actually, this ratio is defined as $r \equiv 16\epsilon_{\text{sr}}$ (which is effectively constant as function of k) and it is constrained to be $\lesssim 10^{-2}$ by CMB B-mode observations.

2.3.3.1 Curvature perturbations in spatially flat slicing

There is a computationally simpler (and more elegant) way to get to the same result as above, although it requires changing gauge. In Newtonian gauge, $\delta\phi$ is coupled to Ψ, hence we can find a gauge in which these perturbations are decoupled. This is the spatially flat slicing gauge, which is a gauge in which the spatial part of the metric is unperturbed. In this gauge the line element is

$$\mathrm{d}s^2 = -(1-A)\mathrm{d}t^2 - 2a\partial_i B \mathrm{d}x^i \mathrm{d}t + a^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j , \tag{2.47}$$

where A and B are the two scalar perturbations. In this gauge, the evolution of $\delta\phi$ is exactly given by Eq. (2.27), since perturbations in the scalar field and in the metric are decoupled. Therefore the power spectrum for the scalar field perturbations found above can be derived without neglecting any metric perturbation.

Now let us find a gauge-invariant variable that is proportional to the scalar field perturbation. We use one of the Bardeen's variables,

$$\mathcal{V} \equiv B + \frac{ik_i}{k^2} \frac{a\delta T^i_0}{\bar{\rho} + \bar{P}} . \tag{2.48}$$

In the Newtonian gauge, the velocity $\boldsymbol{v} = i\boldsymbol{k}\mathcal{V}$ is directly related to this variable. However, in the spatially flat gauge,

$$\mathcal{V} = B - \frac{\bar{\phi}'\delta\phi}{a^2(\bar{\rho} + \bar{P})} \,. \tag{2.49}$$

The Bardeen variable $\Phi^{\rm B}$ defined in the previous chapter is aHB in this gauge. Since the linear combination of two gauge-invariant quantities is still gauge invariant, we can define the curvature perturbation as a gauge-invariant quantity as

$$\mathcal{R} \equiv -\Phi^{\rm B} + aH\mathcal{V} \,, \tag{2.50}$$

which in the the spatially flat gauge, after using the expressions of the mean density and pressure in terms of $\bar{\phi}'$, is $\mathcal{R} = -aH\delta\phi/\bar{\phi}'$, such as we can directly obtain the curvature power spectrum:

$$P_{\mathcal{R}}(k) = \left(\frac{aH}{\bar{\phi}'}\right)^2 P_{\delta\phi}(k) = \left(\frac{2\pi GH^2}{\epsilon_{\rm sr}k^3}\right)_{aH=k} \,, \tag{2.51}$$

which is constant on super-horizon scales at any time. Since the curvature perturbation is a gauge-invariant quantity, the primordial scalar perturbations are usually phrased in terms of its power spectrum. Although we have arrived to its value using a specific gauge, since it is a gauge-invariant quantity, we can relate this power spectrum to the perturbation variables in any gauge. In natural units, the Planck mass $M_{\rm P} = G^{-1/2}$, and let us rephrase

$$P_{\mathcal{R}}(k) = \left(\frac{2\pi H^2}{\epsilon_{\rm sr}M_{\rm P}^2 k^3}\right)_{aH=k} \equiv 2\pi^2 \mathcal{A}_{\rm s} k^{-3} \left(\frac{k}{k_{\rm p}}\right)^{n_{\rm s}-1} \,, \tag{2.52}$$

where $\mathcal{A}_{\rm s}$ is the variance of curvature perturbations in a logarithmic wavenumber interval centered around the pivot scale $k_{\rm p}$ and $n_{\rm s}$ is the scalar spectra index. For the Planck convention, $k_{\rm p} = 0.05\,{\rm Mpc}^{-1}$, $\mathcal{A}_{\rm s} = 2.1 \times 10^{-9}$, which corresponds to a perturbation amplitude $\sim 4.6 \times 10^{-5}$, of similar order of magnitude than the temperature fluctuations in the CMB.

Remember that in the Newtonian gauge $\Phi^{\rm B} = -\Phi$, so the curvature perturbation we found in the previous section is correct. Then,

$$P_{\Phi} = \left(\frac{8\pi GH^2}{9\epsilon_{\rm sr}k^3}\right)_{aH=k} \,, \tag{2.53}$$

as found above.

We can describe the primordial power spectrum from the slope of $k^3 P_{\Phi}$. For instance, if it is constant, it is called a scale-invariant power spectrum. However, there is a small deviation from scale invariance, due to small changes in the slow-roll parameter. The field rolls down the potential slowly in such a way that the Hubble rate, nearly constant, decreases very slowly. This makes that the actual power spectrum is red-tilted, with the larger-scale perturbations (those that leave the horizon earlier) slightly larger than the smaller-scale ones. This feature has been confirmed by CMB observations.

The spectral index can be obtained from the logarithmic derivative of the power spectrum. To obtain it, we need to take the logarithmic derivative of the Hubble rate at horizon crossing:

$$\left(\frac{\mathrm{d}\log H}{\mathrm{d}\log k}\right)_{aH=k} = \frac{k}{H}\frac{\mathrm{d}H}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)_{aH=k} . \tag{2.54}$$

From the definition of the slow-roll parameter in Eq. (2.12) and $\mathrm{d}\tau = -\mathrm{d}(aH)^{-1}$ for a constant Hubble rate,

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)_{aH=k} = -\left(\frac{\mathrm{d}(aH)^{-1}}{\mathrm{d}k}\right)_{aH=k} = k^{-2} , \tag{2.55}$$

we find

$$\left(\frac{\mathrm{d}\log H}{\mathrm{d}\log k}\right)_{aH=k} = -\frac{k}{H}\left(\frac{aH^2\epsilon_{\mathrm{sr}}}{k^2}\right)_{aH=k} = -\epsilon_{\mathrm{sr}} . \tag{2.56}$$

Therefore, if we take the logarithmic derivative of P_Φ,

$$n_{\mathrm{s}} - 1 = \frac{\mathrm{d}}{\mathrm{d}\log k}(\log H^2 - \log\epsilon_{\mathrm{sr}}) \Longrightarrow n_{\mathrm{s}} = 1 - 4\epsilon_{\mathrm{sr}} - 2\delta_{\mathrm{sr}} . \tag{2.57}$$

Analog relations can be obtain for the tensor perturbations. The relation between the scalar and tensor amplitudes and spectral indices are one of the main predictions of inflation (and different inflationary models). Through the impact of these parameters in cosmological observables (and the eventual chance to probe them), we could find a window to probe the Universe at the energy scale of inflation, which could be $\sim 10^{15}$ GeV.

Inflationary perturbations that just enter the horizon today are perceived by us as spatial curvature $\Omega_k \sim (k/\mathcal{H}(\tau_0))^2\mathcal{R}(\boldsymbol{k})$ with $k = H_0$. Therefore, inflation predicts that Ω_k is a random number drawn from a Gaussian distribution with root-mean square $\sim \sqrt{\mathcal{A}_s} \sim 10^{-4}$.

2.3.4 Matter and radiation perturbations

The only piece left is to relate the metric perturbations after inflation to the matter and radiation perturbations. These are the initial conditions for the system of differential equations above. Thanks to the fact that the primordial perturbations in single-field inflation are adiabatic, this derivation is significantly simplified.

We can start by taking the large-scale limit in the Boltzmann equation for the momentum-averaged perturbation of the phase-space distribution of photons from the previous chapter (Eqs. (1.108) and (1.135)). In this limit, $\mathcal{F}'_\gamma \sim \mathcal{F}_\gamma/\tau$, while $ik\mu\mathcal{F}_\gamma \sim k\mathcal{F}_\gamma$: the former is larger than the latter by a $1/k\tau$

factor, which in this limit is very large. This argument allows us to neglect any factor multiplied by k in the Boltzmann equation. Physically, this means that the scales under consideration are much larger than the size of the horizon and therefore are not causally connected. In this regime, only gravity is relevant: dark matter and baryons behave similarly, and their velocities are smaller than overdensity by the same factor $k\tau$. Furthermore, an observer within their causal horizon would only see a uniform sky, so that higher multipoles of the phase space distribution perturbation are negligible. Therefore, we have for radiation (photons and neutrinos alike),

$$\mathcal{F}_{r0} + 4\Phi' = 0\,, \tag{2.58}$$

and for the non-relativistic matter

$$\delta_c' = -3\Phi'\,. \tag{2.59}$$

Note that since we consider adiabatic perturbations and large scales, $\mathcal{F}_{\gamma 0} = \mathcal{F}_{\nu 0}$ and $\delta_c = \delta_b$.

Now we focus on the Einstein equations from Eq. (1.65). The k^2 term can be neglected in this limit and assuming that all energy density is given by radiation (radiation-domination epoch),

$$3\mathcal{H}\left(\Phi' - \mathcal{H}\Psi\right) = 4\pi G a^2 \bar{\rho}_r \mathcal{F}_{r0}\,. \tag{2.60}$$

During radiation domination, $a \propto \tau$, so that $\mathcal{H} = 1/\tau$, and

$$\frac{\Phi'}{\tau} - \frac{\Psi}{\tau^2} = 4\pi G a^2 \bar{\rho}_r \mathcal{F}_{r0} = \frac{\mathcal{F}_{r0}}{2\tau^2}\,, \tag{2.61}$$

where the last equality uses the Friedmann equation. Multiplying by τ^2, differentiating both sides and using $\mathcal{F}_{r0} = -4\Phi'$ we have

$$\Phi''\tau + \Phi' - \Psi' = -2\Phi' \Longrightarrow \Phi''\tau + 4\Phi' = 0\,, \tag{2.62}$$

where the last part neglects anisotropic stress, hence $\Phi = -\Psi$. Inserting the ansatz of $\Phi = \tau^p$ we have

$$p(p-1) + 4p = 0\,, \tag{2.63}$$

which has $p = -3$ and $p = 0$ as solutions. $p = -3$ is a decaying mode, so that it will quickly vanish without contributing to the growth of perturbations. Therefore, we focus on $p = 0$. In this case, from Eq. (2.61), after multiplying by τ^2 and under the same assumptions, for the initial time τ_i

$$\Phi = \frac{\mathcal{F}_{r0}}{2} = \frac{\mathcal{F}_{\gamma 0}}{2} = \frac{\mathcal{F}_{\nu 0}}{2}\,. \tag{2.64}$$

For dark matter and baryons we have $\delta_c(\boldsymbol{k}) = \delta_b(\boldsymbol{k}) = 3\mathcal{F}_{\gamma 0}(\boldsymbol{k})/4 + \text{constant}(\boldsymbol{k})$. Since adiabatic perturbations must have a uniform matter-to-radiation ratio is

$$\frac{n_c}{n_\gamma} = \frac{\bar{n}_c}{\bar{n}_\gamma}\left[\frac{1+\delta_c}{1+3\mathcal{F}_{\gamma 0}/4}\right]\,, \tag{2.65}$$

where the $3/4$ factor for the photon perturbations comes from changing from energy density to number density (at linear order). The combination in the brackets linearizes to $1 + \delta_c - 3\mathcal{F}_{\gamma 0}/4$ must therefore be independent of the position, which forces the constant above to be null for the perturbations to sum up to zero.

From the space-time component of the Einstein equation (Eq. (1.66)) we can get the initial condition for the velocities, using that $\rho_r \gg \rho_m$ and neglecting the k^2 term. We find

$$\mathcal{F}_{\gamma 1} = \mathcal{F}_{\nu 1} = \frac{4\theta_c}{3k} = \frac{4\theta_b}{3k} = -\frac{k}{6aH}\Phi \,, \tag{2.66}$$

which returns the $\mathcal{R} = -3\Psi/2$ we used in the previous section.

In this chapter we have derived the initial conditions for the perturbations, while in the previous chapter we derived their evolution. As a logical progression, in the next chapter we will discuss how perturbations grow, by offering different solutions in specific simplified regimes to the Boltzmann system.

CHAPTER 3

GROWTH OF STRUCTURE

In the previous chapters we have derived the equations describing the evolution of matter, radiation and metric perturbations in the Universe to linear order and the primordial perturbations resulting from inflation. The evolution of perturbations is a coupled system of equations that is usually solved using Boltzmann codes like CLASS or CAMB to quickly compute the cosmological observables. In this chapter we want to get a qualiative understanding of the growth of matter perturbations, which are the ones that will determine the distribution of galaxies in the late Universe.[1]

Since we only attempt to an approximate description of the growth of perturbations, we will reduce significantly the number of equations and limit ourselves to specific limits and regimes. Remember that before recombination, the photon distribution can be characterized by only the monopole and dipole of the momentum-averaged distribution, since all other moments are suppressed due to the tight coupling between photons and baryons. This

[1]This chapter follows *Modern Cosmology* (Ref. [3]) almost in its entirety, but homogenizing nomenclature and altering slightly the order of the discussion.

breaks down after recombination, but at that time the photon perturbations play a negligible role in the growth of structures since the energy-density of the Univese is totally dominated by non-relativistic matter.[2] We will also neglect high multipoles of neutrinos. This is a bad approximation, since neutrinos free stream and are never tightly coupled, but it is better than neglecting them completely. Therefore we will consider the monopole and dipole of the whole relativistic species, photons and neutrinos, all together.

Tight-coupling also allows us to eliminate baryons from the Boltzmann equations, if we are only interested in the qualitative evolution of matter perturbations. This is because the collision term for photons can be neglected in the limit of small baryon density (with respect to photons).[3] Similarly, we will consider than matter perturbations are entirerly determined by cold dark matter.

In this limit, the photon distribution reduces to two equations for the monopole and dipole. Therefore, considering only cold dark matter and total radiation in this limit, we have the following set of differential equations for matter and radiation:

$$\begin{aligned} &\mathcal{F}_{r0}' + k\mathcal{F}_{r1} = -4\Phi' \,, \\ &\mathcal{F}_{r1}' - \frac{k}{3}\mathcal{F}_{r0} = -\frac{4k}{3}\Phi \,, \\ &\delta_c' + \theta_c = -3\Phi' \,, \\ &\theta_c' + \mathcal{H}\theta_c = -k^2\Phi \,. \end{aligned} \tag{3.1}$$

Under these approximations, there is no anisotropic stress, thus $\Phi = -\Psi$ (as used above). Then, we have the time-time component for the Einstein equations (Eq. (1.65)) and the redundant equation from the combination of this one and the time-space components (Eq. (1.66)) to describe metric perturbations and their relations with matter and radiation:[4]

$$\begin{aligned} &k^2\Phi + 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = 4\pi G a^2 (\bar{\rho}_c\delta_c + \bar{\rho}_r\mathcal{F}_{r0}) \,, \\ &k^2\Phi = 4\pi G a^2 \left[\bar{\rho}_c\delta_c + \bar{\rho}_r\mathcal{F}_{r0} + \frac{3\mathcal{H}}{k}\left(\frac{\bar{\rho}_c\theta_c}{k} + \bar{\rho}_r\mathcal{F}_{r1})\right)\right] \,. \end{aligned} \tag{3.2}$$

This set of 5 differential equations is very easy to solve numerically. Analytical solutions are harder to obtain since there is no analytic solution valid on all scales at all times. We need to take limits and specific regimes to study individual pieces of the cosmic evolution and patch them together afterwards.

[2]Following the evolution of the whole photon phase-space distribution is required to understand CMB observations, as primary anisotropies propagate through an evolving Universe, and also to model secondary anisotropies accurately. We will study this problem in the next chapter.

[3]First, since the quadrupole and the polarization are very small, we can neglect the terms multiplying $\mathcal{P}_2$. Then we can show the collision term is proportional to the baryon-to-photon energy ratio $R \equiv 3\bar{\rho}_b/4\bar{\rho}_\gamma$.

[4]Note that only one of them is needed to close the Boltzmann system, since we already fix $\Phi = -\Psi$.

We will study large scales (matter-radiation transition while outside the horizon and horizon crossing during matter domination) and small scales (horizon crossing during radiation-dominated era and matter-radiation transition within the horizon) analytically. We cannot treat analytically modes that enter the horizon around the epoch of equality, and numerical solutions solving the Boltzmann equations are required, but the physics are similar.

In general, we can use the Poisson equation to relate the gravitational potential with the matter perturbations, which is correct for perturbations well within the horizon and in matter domination

$$k^2\Phi = 4\pi G a^2 \bar{\rho}_{\rm m}\delta_{\rm m}\,, \qquad (a \gg a_{\rm eq}, k \gg aH)\,. \tag{3.3}$$

Turning the background matter density using the density parameter and the critical density and the definition of the latter, we can express the matter overdensity as

$$\delta_{\rm m} = \frac{2k^2 a}{3\Omega_{\rm m} H_0^2}\Phi\,, \qquad (a \gg a_{\rm eq}, k \gg aH)\,. \tag{3.4}$$

This kind of conversion will appear many times in this chapter.

The approximations to obtain the equations above are rough. We have neglected the effects of baryons, which are $\sim 16\%$ of matter in the Universe, and the mass of neutrinos (as well as high multipoles of the neutrino and photon perturbations). We will indicate the impact of these additional components as we progress in the chapter.

3.1 Large scales

We can distinguish two different regimes for the large scales. First, the transition from radiation to matter domination takes place while the perturbations are outside the horizon. Second, perturbations enter the horizon already in the matter domination.

3.1.1 Super-horizon solutions

Consider modes far outside the horizon, $k\tau \ll 1$: then we can drop all terms depending on k, which shows that velocities decouple from the system, leaving only three equations to solve.[5] We are left with

$$\begin{aligned} &\mathcal{F}_{r0}' = -4\Phi'\,, \qquad\qquad \delta_c' = -3\Phi'\,,\\ &3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = 4\pi G a^2(\bar{\rho}_c\delta_c + \bar{\rho}_r\mathcal{F}_{r0})\,, \end{aligned} \tag{3.5}$$

with the first two equations showing that the combination $3\delta_c - 4\mathcal{F}_{r0}$ is constant, and zero (since they are adiabatic perturbations, see the discussion

[5]Remember that $\theta = ikv$, hence we also neglect θ terms here.

below Eq. (2.65)). Therefore we drop the equation for radiation. If we now introduce

$$y \equiv \frac{a}{a_{\rm eq}} = \frac{\bar\rho_m}{\bar\rho_r}\,, \tag{3.6}$$

and use it as evolution variable, rather than τ or a.[6] Then, the Einstein equations become

$$\begin{aligned} &3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = 4\pi G a^2 \bar\rho_c \delta_c \left(1 + \frac{4}{3y}\right) \Longrightarrow \\ &y\frac{{\rm d}\Phi}{{\rm d}y} + \Phi = \frac{y}{2(y+1)}\delta_c\left(1+\frac{4}{3y}\right) = \frac{3y+4}{6(y+1)}\delta_c\,, \end{aligned} \tag{3.7}$$

where we have used ${\rm d}/{\rm d}\tau = \mathcal{H} y\,{\rm d}/{\rm d}y$, $a' = a\mathcal{H}$, and the last equality uses the Friedmann equation as function of y. Using the dark-matter equation we have ${\rm d}\delta_c/{\rm d}y = -3{\rm d}\Phi/{\rm d}y$. Then if we express the equation above as an equation for δ_c and derive with respect to y to get ${\rm d}\delta_c/{\rm d}y$ we have

$$\begin{aligned} &-3\frac{{\rm d}\Phi}{{\rm d}y} = \frac{\rm d}{{\rm d}y}\left[\frac{6(y+1)}{3y+4}\left\{y\frac{{\rm d}\Phi}{{\rm d}y} + \Phi\right\}\right] \Longrightarrow \\ &\frac{{\rm d}^2\Phi}{{\rm d}^2 y} + \frac{21y^2+54y+32}{2y(y+1)(3y+4)}\frac{{\rm d}\Phi}{{\rm d}y} + \frac{\Phi}{y(y+1)(3y+4)} = 0\,. \end{aligned} \tag{3.8}$$

Kodama and Sasaki found a solution to this equation in 1984 introducing a new variable

$$u \equiv \frac{y^3}{\sqrt{1+y}}\Phi\,, \tag{3.9}$$

which turns the equation above into

$$\frac{{\rm d}^2 u}{{\rm d}y^2} + \frac{{\rm d}u}{{\rm d}y}\left[-\frac{2}{y} + \frac{3/2}{1+y} - \frac{3}{3y+4}\right] = 0\,, \tag{3.10}$$

where there is no term proportional to u and leaves a first-order equation that is integrable. Denoting here $u' \equiv {\rm d}u/{\rm d}y$ to ease the notation, we have

$$\begin{aligned} &\frac{{\rm d}u'}{u'} = {\rm d}\left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4}\right] \Longrightarrow \\ &\log u' = 2\log y - \frac{3}{2}\log(y+1) + \log(3y+4) + \text{constant} \Longrightarrow \\ &u' = A\frac{y^2(3y+4)}{(1+y)^{3/2}}\,, \end{aligned} \tag{3.11}$$

[6] We could use $\bar\rho_c$ in the numerator to get a slightly more accurate solution, since we are ignoring baryons. But this is not necessary since we are aiming for a qualitative result anyways.

where A is an integration constant to be found. Using the definition of u we can integrate the expression above to get

$$\frac{y^3}{\sqrt{y+1}}\Phi = A\int_0^y \mathrm{d}\tilde{y}\frac{\tilde{y}^2(3\tilde{y}+4)}{(1+\tilde{y})^{3/2}}\,, \tag{3.12}$$

where we have already eliminated the second integration constant since $y^3\Phi \to 0$ as $y \to 0$ (e.g., early times). The second constant can be obtained approximating the integrand in the small y limit, for which we obtain that $\Phi = 4A/3$, thus $A = 3\Phi(0)/4$. The integral above has an analytical solution, which leaves

$$\Phi = \frac{1}{10y^3}\left(16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right)\Phi(0)\,. \tag{3.13}$$

Although it is not obvious, this expression fulfills that at small y, $\Phi = \Phi(0)$. At large y, in turn, once matter dominates, y^3 terms dominates and we find $\Phi = 9\Phi(0)/10$. This means that even at the largest scales, those which never enter the horizon, the gravitational potential drops a factor 9/10 as the Universe undergoes the matter-radiation transition. Remembering that after inflation $\mathcal{R} = 3/2\Phi$, we obtain an important result for super-horizon scales

$$(\Phi(\boldsymbol{k},\tau))_{\mathrm{super-horizon}} = \begin{cases} \frac{2}{3}\mathcal{R}(\boldsymbol{k})\,, & (\text{radiation domination})\,, \\ \frac{3}{5}\mathcal{R}(\boldsymbol{k})\,, & (\text{matter domination})\,. \end{cases} \tag{3.14}$$

We have provided solutions in two limiting times, but the transition between pure radiation and pure matter domination epochs is very long.

Finally this analytic limit solution works reasonably well when compared with numerical results. The main difference is due to the neutrino quadrupole, which introduces a small anisotropic stress and therefore a small slip in the gravitional potentials (i.e., $\Phi \neq -\Psi$). Accounting for this effect drops the 9/10 factor to $\simeq 0.86$.

3.1.2 Horizon crossing

Large scales enter the horizon already in the matter-domination epoch. We have studied their evolution outside the horizon, and now we want to show that also within the horizon the gravitational potential does not evolve over time.

Let us go back to our set of 5 differencial equations from Eqs. (3.1) and (3.2), and focus on scales within the horizon during matter domination. Therefore, we can neglect any role from radiation components, and we keep now the second of the two Einstein equations, which allow us to substitute Φ in the two differential equations for the cold dark matter.

Now we have a set of two differential equations, but we can also add some prior knowledge about the initial conditions: we know that deep in the matter-domination epoch, the gravitational potential on super horizon scales is constant. Therefore, we can set $\Phi' = 0$ as our initial condition. Therefore we need to check if the set of equations admits a solution with constant Φ:

$$\begin{aligned} &\delta_c' + \theta_c = 0\,, \\ &\theta_c' + \mathcal{H}\theta_c = -k^2\Phi\,, \\ &k^2\Phi = \frac{3}{2}\mathcal{H}^2\left[\delta_c + \frac{3\mathcal{H}\theta_c}{k^2}\right]\,, \end{aligned} \tag{3.15}$$

where we have used the Friedmann equation to simplify the last expression. In the matter-dominated era, $H \propto a^{-3/2}$, so that $\mathrm{d}\mathcal{H}/\mathrm{d}\tau = -\mathcal{H}^2/2$. We use the last equation above to obtain δ_c as function of Φ and θ_c and substitute in the first equation, obtaining

$$\frac{2k^2\Phi'}{3\mathcal{H}^2} + \frac{2k^2\Phi}{3\mathcal{H}} - \frac{3\mathcal{H}\theta_c'}{k^2} + \frac{3\mathcal{H}^2\theta_c}{2k^2} + \theta_c = 0\,. \tag{3.16}$$

Now we can use the equation for θ_c' to obtain a second order equation on Φ. We substitute θ_c' above obtaining

$$\frac{2k^2\Phi'}{3\mathcal{H}^2} + \left[\frac{\theta_c}{k^2} + \frac{2\Phi}{3\mathcal{H}}\right]\left(\frac{9\mathcal{H}^2}{2} + k^2\right) = 0\,. \tag{3.17}$$

One condition for constant Φ to be a solution of the system is if we obtain a second-order equation for Φ of the form $\alpha\Phi'' + \beta\Phi' = 0$. Therefore, we can test if Φ constant is a solution by deriving the expression above as function of τ and dropping terms proportional to derivatives of Φ. Using the fact that the conformal time derivative of $\mathcal{H}^{-1}$ is $1/2$ during matter domination and again the equation for θ_c', we see that the remaining terms are

$$-\left[\frac{\mathcal{H}\theta_c}{k^2} + \frac{2\Phi}{3}\right](9\mathcal{H}^2 + k^2) = 0\,, \tag{3.18}$$

where the term in square brackets can be identified with the one in the previous expression, which is proportional to Φ'. Therefore, there is no term proportional Φ and Φ =constant is a valid solution for the system in the matter-domination era. Since it comes also from an initial condition, Φ =constant is *the* solution. The other solution to the system involves a decaying solution, thus not relevant to the problem at hand.

Therefore, gravitational potentials remain constant inside of the horizon during matter-domination era. This means that the matter accretion (which makes the potential grow) and the expansion of the Universe (which dilutes the potential) exactly counteract each other. When dark energy becomes relevant, accelerating the expansion of the Universe, makes the latter dominate and potentials will decay.

In this situation, since the gravitational potential is constant and we are in matter domination and well within the horizon, we can use Eq. (3.4) to relate the potential and the matter perturbations to find that matter perturbations grow as $\propto a$.

3.2 Small scales

We have broken the study of large scales perturbations as the matter-radiation transition outside of the horizon, and the horizon crossing during matter domination. The situation for small scales is mirrored: perturbations enter the horizon during radiation domination, and they experience the transition to matter domination when they are well within the horizon.

3.2.1 Horizon crossing

During radiation domination, matter perturbations are determined by the gravitational potential, but they are not significant to influence it back, since the energy density is dominated by radiation. Therefore, the gravitational potential is influenced by radiation perturbations, and it determines the matter perturbations. The study of the dark matter perturbations in this regime requires a two step process: solve the radiation and potential perturbations, and then translate these into matter perturbations. To start we take the radiation equations in Eq. (3.1) and the second Einstein equation in Eq. (3.2) dropping the matter terms, which leaves

$$\Phi = \frac{3\mathcal{H}^2}{2k^2}\left[\mathcal{F}_{r0} + \frac{3\mathcal{H}}{k}\mathcal{F}_{r1}\right] , \tag{3.19}$$

where as before we have substituted $\bar{\rho}_r$ using the Friedmann equation. Furthermore, in radiation domination, $\mathcal{H} = 1/\tau$, and substituting $\mathcal{F}_{r0}$ by Φ and $\mathcal{F}_{r1}$ using the equation above in the radiation equations we find

$$\begin{aligned} -\frac{3}{k\tau}\mathcal{F}'_{r1} + k\mathcal{F}_{r1}\left[1 + \frac{3}{k^2\tau^2}\right] &= -4\Phi'\left[1 + \frac{k^2\tau^2}{6}\right] - \frac{4k^2\tau}{3}\Phi , \\ \mathcal{F}'_{r1} + \frac{1}{\tau}\mathcal{F}_{r1} &= -\frac{4k}{3}\Phi\left[1 - \frac{k^2\tau^2}{6}\right] . \end{aligned} \tag{3.20}$$

As done before, we will turn these two first-order equations into a second order for Φ. We can use the second equation to express $\mathcal{F}'_{r1}$ as function of Φ and $\mathcal{F}_{r1}$, and substitute in the first equation, which is left as

$$\Phi' + \frac{1}{\tau}\Phi = -\frac{3}{2k\tau^2}\mathcal{F}_{r1} . \tag{3.21}$$

Now we can differentiate, and remove terms depending on $\mathcal{F}_{r1}$ and $\mathcal{F}'_{r1}$ with the expression above. We find

$$\Phi'' + \frac{4}{\tau}\Phi' + \frac{k^2}{3}\Phi = 0 , \tag{3.22}$$

which is the wave equation in Fourier space with a damping term due to the expansion of the Universe. This implies oscillatory solutions, which must be connected to the initial condition of a constant Φ (before horizon crossing). Therefore, let us define $u \equiv \Phi\tau$, such as

$$u'' + \frac{2}{\tau}u' + \left(\frac{k^2}{3} - \frac{2}{\tau^2}\right)u = 0\,. \tag{3.23}$$

This is the Bessel equation of order 1, with solutions $j_1(k\tau/\sqrt{3})$ (the spherical Bessel function) and $n_1(k\tau/\sqrt{3})$ (the spherical Neumann function). The latter diverges as $\tau \to 0$, so that we must discard it due to the initial conditions. We can use the exact expression for $j_1(x) = (\sin x - x\cos x)/x^3$, which tends to $1/3$ as $x \to 0$. Since $\Phi(0) = 2\mathcal{R}/3$, we obtain

$$\Phi(\boldsymbol{k}, \tau) = 2\frac{j_1(k\tau/\sqrt{3})}{k\tau/\sqrt{3}}\mathcal{R}(\boldsymbol{k})\,. \tag{3.24}$$

As soon as the mode enters the horizon during radiation-dominated era, its potential starts to decay and oscillate. Effectively, the solution corresponds to a damped standing wave in Fourier space. Physically, this is because radiation pressure counteracts (and overcomes) gravity, preventing overdensities to grow. This is evident from Eq. (3.19), ignoring the dipole (which is much smaller than the monopole within the horizon): since $\mathcal{F}_{r0}$ oscillates with fixed amplitude, the potential also oscillates but proportionally to $\mathcal{H}^2 \propto \tau^{-2}$.

Neglecting the influence of dark matter induces an error in the evolution of the gravitational potential at large scales, due to its gravitational effect. The effect of free-streaming neutrinos leads to additional damping of the potential after horizon crossing.

Now we can determine the evolution of the cold dark matter perturbations, which are determined by Φ, following Eq. (3.1). Merging both equations, we find (using that $\mathcal{H} = 1/\tau$ in radiation domination)

$$\delta_c'' + \frac{1}{\tau}\delta_c' = S = -3\Phi'' + k^2\Phi - \frac{3}{\tau}\Phi'\,. \tag{3.25}$$

Two solutions to the homogeneous equation (i.e., having the source term $S = 0$) are δ_c =constant and $\delta_c = \log\tau$. Therefore, we anticipate a logarithmic growth of the matter perturbations within the horizon in the radiation-dominated epoch.

Remember that the solution to a second-order equation is the linear combination of the two homogeneous solutions and a particular solution. In this case, we do not have prior intuition about the particular solution, so we can construct it from the two homogeneous solutions (denoted by s_1 and s_2) and the source term. Such solution is the integral of the source term weighted by the Green function $[s_1(\tau)s_2(\tilde{\tau}) - s_1(\tilde{\tau})s_2(\tau)]/[s_1'(\tilde{\tau})s_2(\tilde{\tau}) - s_1(\tilde{\tau})s_2'(\tilde{\tau})]$. So here we have (adding factor of k to the arguments of the logarithms, since they will be convenient later)

$$\delta_c = C_1 + C_2\log(k\tau) - \int_0^\tau \mathrm{d}\tilde{\tau} S(k,\tilde{\tau})\tilde{\tau}(\log(k\tilde{\tau}) - \log(k\tau))\,. \tag{3.26}$$

At very early times, the integral can be neglected, and matching the initial condition ($\delta_c = \mathcal{R}$, from previous chapter), we find $C_2 = 0$ and $C_1 = \mathcal{R}$. S decays as it enters the horizon (since the potential does), hence most of the contribution to the integral comes from $k\tau \sim 1$. Therefore, the first integral will asymptote to a constant, and the second one will lead to a term proportional to $\log(k\tau)$. Therefore, after entering the horizon

$$\delta_c = A\mathcal{R}\log(Bk\tau)\,, \tag{3.27}$$

which is a constant plus a logarithmic growing mode. The constant term is C_1 plus the first integral, while the logarithmic term is the second integral:

$$\begin{aligned} A\mathcal{R}\log B &= \mathcal{R} - \int_0^\infty \mathrm{d}\tilde{\tau} S(k,\tilde{\tau})\tilde{\tau}\log(k\tilde{\tau})\,, \\ A\mathcal{R} &= \int_0^\infty \mathrm{d}\tilde{\tau} S(k,\tilde{\tau})\tilde{\tau}\,. \end{aligned} \tag{3.28}$$

The upper limit set to infinity is allowed since the potential decays (and thus S) and the integrand vanishes at large τ. Solving this equations return $A = 6$ and $B = 0.44$. A more precise treatment, using more precise expressions for the potentials, leads to slightly different values, as found by Hu and Sugiyama in 1996.

In summary, dark matter perturbations grow even during radiation-domination era. This is in contrast of the radiation perturbations, which oscillate with constant amplitude (determining the decay of the potential) and the baryon perturbations, which are tightly coupled to photons. This is because cold dark matter does not feel any pressure that counteracts the effect of gravity, hence even if the gravitational potential decays and the Universe expands faster it keeps clustering (although not as fast as during matter domination era, where the constant potential implied $\delta_c \propto a$). As the Universe gets closer to matter domination, the expansion slows down and the perturbations start to grow faster. This growth eventually makes that the matter perturbations must be taken into account (i.e., $\bar{\rho}_c\delta_c \sim \bar{\rho}_r\mathcal{F}_{r0}$), which produces the small offset at large scales in our prediction for the gravitational potential inside the horizon during radiation domination mentioned before.

3.2.2 Sub-horizon evolution across the matter-radiation transition

As we mentioned, even during radiation domination, the growth of matter perturbations joint to the fact that the radiation perturbations oscillate at fixed amplitude eventually leads to $\bar{\rho}_c\delta_c \sim \bar{\rho}_r\mathcal{F}_{r0}$ even if $\bar{\rho}_c < \bar{\rho}_r$. Once this point is reached, the gravitational potential is determined by the matter perturbations independently of the radiation perturbations. Therefore, $\mathcal{F}_r$ can be ignored. In this subsection we will solve the evolution of perturbations in this regime and match it to the logarithmic growth from the previous subsection, which happened when the potential decays.

We start from Eq. (3.1), neglecting the role from radiation in this case, and the second Einstein equation in Eq. (3.2), and once again we want to get to a second order equation from a system of three equations. In this regime, the sub-horizon dark-matter perturbations experience the matter-radiation transition, so we will use again the variable y defined in Eq. (3.6) as the evolution variable. The three equations therefore become

$$\begin{aligned}\frac{\mathrm{d}\delta_c}{\mathrm{d}y}+\frac{\theta_c}{\mathcal{H}y}&=-3\frac{\mathrm{d}\Phi}{\mathrm{d}y}\,,\\ \frac{\mathrm{d}\theta_c}{\mathrm{d}y}+\frac{\theta_c}{y}&=-\frac{k^2\Phi}{\mathcal{H}y}\,,\\ k^2\Phi&=\frac{3\mathcal{H}y}{2(y+1)}\delta_c\,.\end{aligned}\tag{3.29}$$

Note that expressed in this way the gravitational potential only depends on δ_c and not on the velocity divergence because perturbations are well within the horizon and terms that are divided by $\mathcal{H}/k \ll 1$ (remember that $\theta = ikv$). Following the same routine as above, we differentiate the first equation above to get

$$\frac{\mathrm{d}^2\delta_c}{\mathrm{d}y^2}-\frac{(2+3y)\theta_c}{2\mathcal{H}y^2(1+y)}=-3\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2}+\frac{k^2\Phi}{\mathcal{H}^2y^2}\,,\tag{3.30}$$

where we have used the second equation above to substitute the derivative of θ_c, and considered that $\mathrm{d}(\mathcal{H}y)^{-1}/\mathrm{d}y = -(1+y)^{-1}(2\mathcal{H}y)^{-1}$. The first term in the right is much smaller than the second one, which has a $k^2/\mathcal{H}^2$ factor, hence we drop it, and we can substitute the second term using the Einstein equation above. Using the first equation for δ_c we can substitute the θ_c factor (neglecting the potential, which is much smaller than δ_c within the horizon, according to the Poisson equation). Thus, we have

$$\frac{\mathrm{d}^2\delta_c}{\mathrm{d}y^2}+2\frac{(2+3y)}{2\mathcal{H}y^2(1+y)}\frac{\mathrm{d}\delta_c}{\mathrm{d}y}-\frac{3}{2y(y+1)}\delta_c=0\,,\tag{3.31}$$

which is known as the Meszaros equation, and governs the evolution of sub-horizon cold dark matter perturbations after radiation perturbations have become negligible.

Now we need to find the two solutions and match the to the logarithmic evolution found above. We can use our prior knowledge about the perturbations deep in the matter era, which we have seen they grow proportionally to a. Therefore, one of the solutions must be a polynomial of y of order 1 (which would imply $\mathrm{d}^2\delta_c/\mathrm{d}y^2 = 0$). In this case,

$$\frac{\mathrm{d}\delta_c}{\mathrm{d}y}\frac{1}{\delta_c}=\frac{3}{2+3y}\,,\tag{3.32}$$

the solution of which is $\delta_c \propto y + 2/3$, or

$$\delta_c \propto a+\frac{2a_{\mathrm{eq}}}{3}\,,\tag{3.33}$$

which approximates to the growth proportional to a for $a \gg a_{\rm eq}$.

The second solution can be found using $u \equiv \delta_c/(y+2/3)$, which satisfies

$$(1+3y/2)\frac{{\rm d}^2u}{{\rm d}y^2} + \frac{(21/4)y^2+6y+1}{y(y+1)}\frac{{\rm d}u}{{\rm d}y} = 0\,, \tag{3.34}$$

and involves a first-order equation in the derivative of u. We can therefore integrate to get the solution for this derivative, and then integrate again. The first integral returns

$$\frac{{\rm d}u}{{\rm d}y} \propto (y+2/3)^{-2}y^{-1}(y+1)^{-1/2}\,, \tag{3.35}$$

and the subsequent integral leads to

$$\delta_c \propto (y+2/3)\log\left[\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right] - 2\sqrt{1+y}\,. \tag{3.36}$$

At early times $y \ll 1$, the first solution is constant, and the second, proportional to $\log y$; at late times $y \gg 1$, the first solution scales as y and the second decays as $y^{-3/2}$. Therefore, we can denote them as the growing D_+ and decaying D_- modes, respectively.

Note that the decaying mode cannot be neglected because we need to match the solution to the logarithmic evolution from horizon crossing derived in the previous subsection, which is valid within the horizon before equality. Therefore, we can aspire to get a qualitative solution only for the modes that enter the horizon before equality.

For those modes we can match the two solutions and their first derivatives (with respect to y),

$$\begin{aligned} A\mathcal{R}\log(By_m/y_H) &= C_1D_+(y_m) + C_2D_-(y_m)\,,\\ \frac{A\mathcal{R}}{y_m} &= C_1D'_+(y_m) + C_2D'_-(y_m)\,, \end{aligned} \tag{3.37}$$

where y_m is the matching time, which must satisfy $y_H \ll y_m \ll 1$, and y_H is the horizon crossing time, which replaces $k\tau$ in the logarithm with y/y_H, valid as long y_m is deep in the radiation era. As you can see, at late times, the only term that matters is D_+, since D_- decays with time.

3.3 Transfer function

We have seen that at linear order each mode $\boldsymbol{k}$ evolves independently from the rest for all species and metric perturbations. Furthermore, while the initial conditions are random given a distribution function, the linear evolution is deterministic. Therefore, we can express any property of a field as function of its initial condition using a transfer function T_X. In the absence of anisotropic stress (e.g., as the one introduced by the massive neutrinos), the transfer function can be decomposed in the time and k dependence $T(a, k) = T(k)D(a)$, where D is known as the linear growth factor, and will be discussed later. Since the effect of neutrinos is not large, and its anisotropic stress is small, there is only a small scale-dependence on the growth factor. However, as we will discuss later, it is key to accurately describe the growth of perturbations.

Let us focus on a specific flavor of the transfer function, regarding the relation between the gravitational potential at a given time with respect to the large-scale beyond-horizon gravitational potential during matter domination (i.e., after accounting for the reduction by the 9/10 factor). Let us also consider that we can completely separate the scale and time dependence on the transfer factor into two different multiplicative factors (the transfer function and the growth factor).[7] Therefore, we can write

$$\Phi(\boldsymbol{k}, a) = \frac{3}{5}\mathcal{R}(\boldsymbol{k})T(k)\frac{D_+(a)}{a}\,, \tag{3.38}$$

where the prefactor accounts for the 9/10 factor of super-horizon scales after matter-radiation equality. The normalization of the growth factor D, although seemingly strange, is like that because it is defined in term of the matter perturbations during matter domination, rather than the gravitational potential.

Using the Poisson equation (in matter domination, well within the horizon), and using the matter density parameter and the definition of the critical density as we have done many times in this chapter, we have (in this limit)

$$\delta_m(\boldsymbol{k}, a) = \frac{2k^2 a}{3\Omega_m H_0^2}\Phi(\boldsymbol{k}, a) = \frac{2k^2}{5\Omega_m H_0^2}\mathcal{R}(\boldsymbol{k})T(k)D_+(a)\,, \tag{3.39}$$

which by definition implies that the time evolution of matter perturbation is linearly proportional to the growth factor. Note that this definition of the transfer function can be extended to any variable, especially if we define in full generality

$$\delta_x(\boldsymbol{k}, a) = \mathcal{R}(\boldsymbol{k})T(k, a)\,. \tag{3.40}$$

As we saw before, modes that enter the horizon after matter-radiation equality have a constant potential. Therefore, the transfer function is very close to unity at scales beyond the size of the horizon at matter-radiation equality, those that fulfill $k \ll k_{\rm eq} = \mathcal{H}_{\rm eq}$. For the consensus cosmology, $k_{\rm eq} = 0.073\,{\rm Mpc}^{-1}\Omega_m h^2 = 0.010\,{\rm Mpc}^{-1}$.

Now, recovering the definition of the power spectrum of primordial curvature perturbations from Eq. (2.51), we find that the linear matter power

[7]Massive neutrinos and, in more generality, a non-negligible anisotropic stress introduces a scale dependence in the time evolution of the matter perturbations, which breaks down this assumption.

spectrum is given by

$$P(k,a) = \frac{8\pi}{25}\frac{\mathcal{A}_s}{\Omega_m^2 H_0^4} T^2(k) D_+^2(a) \frac{k^{n_s}}{k_{\rm p}^{n_s-1}}\,. \tag{3.41}$$

The power spectrum is the Fourier transform of the correlation function $\langle \delta(\boldsymbol{x})\delta(\boldsymbol{x}')\rangle$, hence it must have units of volume; we can see in the expression above that this is fulfilled.

To get an analytic expression for the transfer function in this limit, we can recover the results from Eq. (3.37) and get the value for the constant multiplying the growing mode:

$$C_1 = \frac{D'_-(y_m)\log(By_m/y_H) - D_-(y_m)/y_m}{D_+(y_m)D'_-(y_m) - D'_+(y_m)D_-(y_m)} A\mathcal{R}\,. \tag{3.42}$$

The denominator is $-(4/9)y_m^{-1}(y,+1)^{-1/2} = -4/9y_m$, since $y_m \ll 1$. In that limit, $D_- \to (2/3)\log(4/y) - 2$ and $D'_- \to -2/3y$, so that

$$C_1 \to -\frac{9}{4}A\mathcal{R}\left[-\frac{2}{3}\log(By_m/y_H) - (2/3)\log(4/y_m) + 2\right]\,, \tag{3.43}$$

which happens to not depend on y_m. This returns an approximate solution at late times for the small-scale dark matter perturbations in our simplified scenario:

$$\delta_c(\boldsymbol{k},a) = \frac{3}{2}A\mathcal{R}(\boldsymbol{k})\log\left(\frac{4Be^{-3}a_{\rm eq}}{a_H}\right) D_+(a)\,, \qquad (a \gg a_{\rm eq})\,, \tag{3.44}$$

where a_H is the scale factor at which the mode k enters the horizon, $a_H H(a_H) = k$. For very small scales, the argument of the logarithm simplifies, since $a_{\rm eq}/a_H \to \sqrt{2}k/k_{\rm eq}$ (due to the time dependence of the Hubble rate during matter domination). Then, the transfer function (in this limit in which we have ignored baryons and anisotropic stress) is given by

$$T(k) = \frac{15}{4}\frac{\Omega_m H_0^2}{k^2 a_{\rm eq}} A \log\left(\frac{4Be^{-3}\sqrt{2}k}{k_{\rm eq}}\right)\,, \qquad (k \gg k_{\rm eq})\,. \tag{3.45}$$

Plugging the numbers, we have

$$T(k) = 12\frac{k_{\rm eq}^2}{k^2}\log(0.12k/k_{\rm eq})\,, \qquad (k \gg k_{\rm eq})\,. \tag{3.46}$$

This approximation is valid at $k \gtrsim 1\,{\rm Mpc}^{-1}$. There have been derivations with more accurate analytic solutions, but since Boltzmann codes have become so fast and precise, they have lost most of their practical utility by now, beyond providing some qualitative understanding of the evolution of perturbations.

If there had been no logarithmic growth of the matter perturbations during radiation domination, the modes that entered the horizon before equality would have not growth until the epoch of equality, having their amplitude suppressed with respect to large-scale modes by a factor of order $(k_{\rm eq}/k)^2$ (instead having also the logarithmic factor).

We have now the tools to qualitative explain some of the features of the matter power spectrum. In the power spectrum we find a clear turnover scale at $k_{\rm eq}$. Larger scales enter the horizon after equality, hence they have had a constant potential over all their evolution (approximately). This makes that the transfer function at those scales is approximately unity, and the matter power spectrum to be $\propto k$ (accounting for the k^2 relation between δ_c and Φ and the scale dependence of the primordial power spectrum). Smaller scales, however, enter the horizon at earlier times, during the radiation-domination era, and have the potential suppressed. Although this still implies a logarithmic growth for the matter perturbations, they are suppressed by a factor $\sim (k_{\rm eq}/k)^2 \log(0.12k/k{\rm eq})$, and therefore the power spectrum decreases with k.

If we keep zero curvature and h fixed, changes in Ω_m change the position and amplitude of the turnover ($k_{\rm eq} \propto \Omega_m h^2$ in physical units, $\propto \Omega_m h$ in Mpc$/h$ units): for lower abundance of matter, equality happens later and $k_{\rm eq}$ is smaller, and viceversa.

3.3.0.1 Effect of baryons and massive neutrinos

After equality, the solution of Eq. (3.37) is not accurate due to the impact of baryons. Baryons contribute after equality to the gravitational potential, but they cluster less than dark matter due to the radiation pressure that they feel until recombination. This solution therefore overestimates the growth of matter perturbations. In a more realistic scenario, baryons suppress matter overdensities in scales below the size of the horizon at equality, given by $k_{\rm eq} \sim 0.01\,{\rm Mpc}^{-1}$ in the fiducial cosmology.

There is another big impact of baryons in the matter perturbations that we have not considered. Before decoupling, the baryon-photon fluid experiences acoustic oscillations (due to the counteracting forces of the radiation pressure and gravity). We saw similar acoustic oscillations in the potential in the radiation-dominated era. Those oscillations reflect the oscillations in the density of the baryon-photon fluid, which are known as baryon acoustic oscillations. The amplitude of the oscillations is small due to the relative abundance of baryons with respect to the total matter.

Massive neutrinos affect the expansion in the Universe (as they become non relativistic), although this does not affect the moment of equality because the non-relativistic transition happens at $z \sim 100$. However, even if non-relativistic, they do free stream, i.e., they are not cold, as dark matter and baryons. Therefore, they travel across perturbations diluting them in scales below the free-streaming scale (determined by the comoving distance a massive neutrino can travel in a Hubble time):

$$k_{\rm fs}(a) \simeq 0.063\, h\, {\rm Mpc}^{-1} \frac{m_\nu}{0.1\,{\rm eV}} \frac{a^2 H(a)}{H_0}\,. \tag{3.47}$$

Therefore, the presence of massive neutrino suppresses the power spectrum at $k \gtrsim k_{\rm fs}$, in a scale-dependent time-dependent suppression, since the free-streaming scale (and the level of suppression) depends on time. The suppressing factor with respect to the massless neutrinos case at small scales asymptotes to a constant. More massive neutrinos suppress more than lighter, but at smaller scales (since their free streaming scale is smaller) and viceversa. This means that at large scales, the perturbations for more massive neutrinos may be larger than for lighter neutrinos.

After recombination, free of the radiation pressure, baryons eventually follow the dark matter distribution as they fall in its potential wells, and follow the matter equations in Eq. (3.1). Let us define the relative density perturbation and the relative velocity between baryons and dark matter:

$$\delta_{bc} = \delta_b - \delta_c \,, \qquad v_{bc} = v_b - v_c \,, \qquad \theta_{bc} = \theta_b - \theta_c \,. \tag{3.48}$$

Their evolution equations can be obtained from substracting the evolution equations of each component, yielding

$$\delta'_{bc} + \theta_{bc} = 0 \,, \qquad\qquad \theta'_{bc} + \mathcal{H}\theta_{bc} = 0 \,. \tag{3.49}$$

There is no impact of the gravitational potential here, because the gravitational potential cares only about the total matter. The solutions for the system above involves a solution with constant relative density perturbations and no relative velocity, and a decaying mode for the total relative velocity $\theta_{bc} \propto a^{-1}$, with $\delta_{bc} \propto \int \mathrm{d}\tau/a$. The latter corresponds to giving baryons an initial push such as they have a different initial condition than dark matter. This is actually the realistic case, since after recombination, baryons have a different velocity than dark matter as they fall in its potential wells. Nonetheless, this difference in the state after recombination is washed out by the gravitational pull of dark matter by the time we observe the large-scale structure.

It is relevant, nonetheless, for the early time perturbations at very small scales: after recombination, v_{bc} is supersonic, which means that baryons can travel over dark matter potential wells diluting them rather than actually falling in them. This is why for early times it is necessary to study the small-scale limit as function of a bulk relative velocity between the two species. The variance of such bulk velocity is determined by the physics of the photon-baryon plasma before recombination. The main impact is that, at early times, the supersonic bulk relative velocity suppresses the growth of structures at very small scales, with different patches of the Universe showing different levels of suppression that are correlated at large distances following the baryon acoustic oscillations pattern.

3.3.1 Growth factor

We can also discuss the time evolution of the matter perturbations, in terms of a scale-independent linear growth factor. At late times, the horizon is much larger than the scales of interest, and the only deviation that we find from the Meszaros equation is the influence of dark energy. Furthermore, at these times (after recombination), baryons do not feel any pressure and therefore behave like cold dark matter (except for the acoustic term that matters at very small scales). While dark matter and baryons start with different initial conditions after recombination, baryons fall in the dark matter potential wells and trace the dark matter perturbations faithfully. Therefore, we will use the total matter perturbations (with a energy-density weighted average).

We start from the matter equations in Eq. (3.1), multiplying the first one by a a deriving with respect to the conformal time. Neglecting the second derivative of Φ, since it is negligible within the horizon, we have

$$(a\delta_m')' = ak^2\Phi\,, \tag{3.50}$$

which we can combine with the Einstein equation of Eq. (3.2). Neglecting contributions from radiation and terms that are small when $k \gg \mathcal{H}$ and using the Friedmann equation and the density parameter, we have

$$(a\delta_m')' = \frac{3}{2}\Omega_m H_0^2 \delta_m\,. \tag{3.51}$$

To solve this equation it is better to use a as the time variable, which returns

$$\frac{\mathrm{d}^2\delta_m}{\mathrm{d}a^2} + \frac{\mathrm{d}\log(a^3H)}{\mathrm{d}a}\frac{\mathrm{d}\delta_m}{\mathrm{d}a} - \frac{3\Omega_m H_0^2}{2a^5H^2}\delta_m = 0\,, \tag{3.52}$$

which has to be solved numerically. We can use the variable $u = \delta_m H^{-1}$, that leaves the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}a^2} + 3\left[\frac{\mathrm{d}\log H}{\mathrm{d}a} + \frac{1}{a}\right]\frac{\mathrm{d}u}{\mathrm{d}a} = 0\,. \tag{3.53}$$

The first order equation can be integrated to obtain $\mathrm{d}u/\mathrm{d}a \propto (aH)^{-3}$. If we integrate again, and remembering that the growth factor is uH, we have

$$D_+(a) \propto H(a)\int^a \frac{\mathrm{d}a'}{(a'H(a'))^3}\,. \tag{3.54}$$

Now we need to find the normalization. We can find it matching the behavior of the definition of the growth factor $D_+(a) = a$ during matter domination. Therefore, since at those times $H = H_0\sqrt{\Omega_m a^{-3}}$,

$$D_+(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0}\int_0^a \frac{\mathrm{d}a'}{(a'H(a')/H_0)^3}\,. \tag{3.55}$$

This is only valid for matter and a cosmological constant components.

We can find the solution for the decaying mode assuming $\delta_m = H$, expressing the equation in terms of H^2 and substituting H^2 for the sum of the components and their evolution. In this case, we will find the condition that for $\delta_m = H$ to work as a solution, any component beyond matter must fulfill $p_s^2 + 2p_s = 0$, which is the same condition for the growing mode.

Finally, a relevant quantity for large-scale structure is the logarithmic derivative of the growth factor, known as the growth rate f, defined as

$$f(a) \equiv \frac{\mathrm{d}\log D_+}{\mathrm{d}\log a} \simeq (\Omega_m(a))^{0.55}\,, \tag{3.56}$$

where the last equality involves a fitting function which depends on the time-dependent matter density parameter. The growth rate reduces to $f = 1$ in the totally matter dominated Universe (i.e., $\Omega_m = 1$), and it is only when dark energy becomes relevant that the growth factor over the scale factor (D/a) (and also f) start to decay.

Before closing this chapter, let us note that there is a slightly different convention regarding the normalization of the growth factor. It can be defined in terms of early-time perturbations, as we have done so far. However, for studies of large-scale structure, it is more common to find it defined in terms of the matter power spectrum in the present day. In that case, the growth factor would fulfill

$$P_m^2(k,a) = D_{\mathrm{LSS}}^2 P_m^2(k,a_0)\,. \tag{3.57}$$

Of course, these two conventions only differ in their normalization.

3.4 Limit of linear theory

We have limited the discussion to linear perturbations so far. Non linearities, which significantly complicate the study of the growth of perturbations and large-scale structure, are bound to be relevant at small scales. There are different ways to estimate the scales at which non linearities cannot be ignored. One of them is to compute the variance of linear perturbations in a certain spatial scale. For instance, consider an spherical top-hat region in Fourier space (which corresponds to a sinc window function in configuration space, and viceversa), and the variance will be given by

$$\sigma_{\mathrm{w}}^2 = \frac{1}{2\pi^2}\int \mathrm{d}k k^2 W^2(k) P(k)\,, \tag{3.58}$$

where $P(k)$ is the linear power spectrum and $W(k)$ is the spherically symmetric window function in Fourier space of the region we are considering. If $\sigma^2 \gtrsim 1$, the perturbations are too large for the linear regime to accurately describe them, and non-linear growth is relevant for the study. We can therefore scan σ_{w}^2 as function of radius (or scale) to find the scale k_{NL} at which non-linear perturbations become relevant.

Another way to estimate $k_{\rm NL}$ is to consider the variance of modes within a specific narrow logarithmic wavenumber:

$$\sigma_{\rm L}^2 = \frac{1}{\epsilon}\int_{|\log k' - \log k|<\epsilon} \frac{{\rm d}\Omega_{\rm k}{\rm d}\log k k^3}{(2\pi)^3} P(k) = \frac{k^3}{2\pi^2} P(k)\,, \tag{3.59}$$

where for the last equality we have assumed an infinitesimal wavenumber bin. Similarly, linear perturbations fulfill $\sigma_{\rm L}^2 \ll 1$, while values close tu unity indicate non-linear perturbations. Today, this corresponds to a non-linear scale of $k_{\rm NL}(a=1) \simeq 0.25\,h\,{\rm Mpc}^{-1}$, and progressively higher values as we go higher in redshift (since structure did not have time to grow so much).

CHAPTER 4

COSMIC MICROWAVE BACKGROUND

In the previous chapter we studied, under some simplifying assumptions and in specific limits, how the gravitational potential evolved and how this impacted the dark matter perturbations. The matter distribution in the Universe is relevant because it is the one that determines the potential wells in which galaxies will form, and make up for the large-scale structure we observe in the Universe today. However, we did not pay much attention to the photon perturbations. Given how precise the observations of the cosmic microwave background (CMB) anisotropies are, understanding photon perturbations and predicting them accurately is crucial to understand our Universe and constrain cosmological models.

As expected, the photon perturbations behave drastically different before and after recombination, which takes places around $z_* \sim 1100$.[1] Before recombination, the interactions between photons and free electrons are so frequent that photons and baryons are tightly coupled and can be described as a single fluid; after recombination, in turn, photons free stream from the

[1] We will denote quantities related with recombination with a subscript '*'.

last-scattering surface. Since gravitational potentials are too weak to trap photons, photon overdensities do not grow after recombination, contrary to dark matter and baryons.

As discussed earlier, we can describe photons in terms of the perturbations in their phase-space distribution. In Eq. (1.75) we defined the perturbations in the phase-space distribution as $f(x^i, P_j, \tau) = f_0(q, m)(1 + \varphi(x^i, q, \hat{\boldsymbol{q}}_j, \tau))$, which in the case of the photons can be further simplified taking the momentum-averaged perturbation $\mathcal{F}_\gamma$. In Fourier space, the momentum-averaged perturbation of the phase space distribution only depends on the Fourier mode, direction of the momentum, and conformal time, and it can be related to the photon overdensity, velocity divergence and anisotropic stress following Eq. (1.107).

Nonetheless, we cannot measure directly those photon properties. In turn, we can measure the intensity of the radiation that arrives along a given line of sight as function of frequency (and the polarization of that radiation). Therefore, rather than dealing with the photon properties, it is more convenient to work with the temperature T that determines its background phase-space distribution

$$f_0 = f_0(\epsilon) = \frac{g_*}{h_P^3} \frac{1}{\exp\{\epsilon/k_{\rm B}T_0\} \pm 1}\,, \tag{4.1}$$

where as in the derivation of the Boltzmann equations we use $\epsilon = aE = a\sqrt{p^2 + m^2} = \sqrt{P^2 + a^2 m^2}$ and $T_0 = aT$ as the temperature of the particles today. At linear order, perturbations in the photon distribution maintain the black-body spectrum, but change the associated temperature of the distribution. Hence, we can equally describe the perturbations in the phase space distribution with perturbations in the temperature:

$$T = \bar{T}(1 + \Theta) \implies \Theta = \frac{T - \bar{T}}{\bar{T}}\,. \tag{4.2}$$

Therefore, if we substitute this expression for the temperature in f_0, we find that $f = f_0(q/(1 + \Theta))$ in such a way that at linear order[2]

$$\Theta = -\left(\frac{\mathrm{d}\log f_0}{\mathrm{d}\log q}\right)^{-1} \varphi = \frac{1}{4}\mathcal{F}_\gamma\,. \tag{4.3}$$

We can observe the photons from the last-scattering surface. Therefore, we can only study the CMB as function of position on the sky, not in terms of any radial distance. This is why we will focus on angular summary statistics to describe the angular maps obtained from the CMB observations. In particular, we will focus on the angular power spectrum. To do that, let us define the

[2]Remember that since photons are massless, $\epsilon = q$, where $q = ap$ is the comoving momenta used in the chapter about cosmological perturbation evolution.

temperature perturbation as function of a three-dimensional position,

$$\begin{aligned}\Theta(\boldsymbol{x},\hat{\boldsymbol{q}},\tau) &= \int \mathrm{d}^3k e^{i\boldsymbol{k}\boldsymbol{x}}\Theta(\boldsymbol{k},\hat{\boldsymbol{q}},\tau) = \\ &= \int \mathrm{d}^3k e^{i\boldsymbol{k}\boldsymbol{x}} \sum_{\ell=0}^{\infty}(-i)^\ell(2\ell+1)\Theta_\ell(\boldsymbol{k},\tau)\mathcal{P}_\ell(\mu)\,,\end{aligned} \tag{4.4}$$

where $\mu = \hat{\boldsymbol{k}}\hat{\boldsymbol{q}}$ is the cosine of the angle between $\boldsymbol{k}$ and the propagation direction of the photone. Note that the direction of the momentum of the photon must be the same as the direction in which an observer at the origin (us) looks at the sky to detect them (with a different sign). Therefore, let us change $\hat{\boldsymbol{q}}$ for the angle on the sky $\hat{\boldsymbol{n}}$. The anisotropy at the origin as function of position on the sky is

$$\begin{aligned}\Theta(\hat{\boldsymbol{n}}) &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}Y_{\ell m}(\hat{\boldsymbol{n}})\,,\\ a_{\ell m}(\boldsymbol{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\int \mathrm{d}\Omega_{\mathrm{n}}Y^*_{\ell m}(\hat{\boldsymbol{q}})\Theta(\boldsymbol{k},\hat{\boldsymbol{q}},\tau)\,,\end{aligned} \tag{4.5}$$

where we have used the orthonormality of the spherical harmonics.

We cannot make any predicction about specific values of the perturbations in a specific point (or a specific coefficient $a_{\ell m}$ in this case); we can only predict their ensemble average, which is measured in practice using the Ergodic hypothesis. The covariance of the expansion coefficients $a_{\ell m}$ is given by the angular power spectrum:

$$\langle a_{\ell m}a^*_{\ell' m'}\rangle = C_\ell\delta_{\ell\ell'}\delta_{mm'}\,. \tag{4.6}$$

The relation between the angular power spectrum and the angular 2-point correlation function $w(\theta)$ for two points separated by an angle θ which fulfills $\cos\theta = \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}'$ is

$$w(\theta) = \left\langle\Theta(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}')\right\rangle = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)C_\ell\mathcal{P}_\ell(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}')\,. \tag{4.7}$$

Excluding the monopole and dipole (i.e., for $\ell \geq 2$), the power spectrum and correlation function are gauge-independent quantities.

Furthermore, note that the photons we see today had to travel off the potential they were at the last-scattering surface, which changes their energy accordingly to the sign of the potential: they lose energy if they were in an overdensity ($\Psi < 0$), and viceversa, due to the gravitational redshift. Therefore, the actual observed temperature is $\Theta_0 + \Psi_*$.

As we have already mentioned, there are two opposing forces influencing the photon-baryon fluid. On the one hand, there is gravity, for which the potential wells in the dark matter overdensity pull the fluid in. On the other, radiation pressure between photons and baryons grows with density, diluting the photon-baryon overdensities and therefore pushing against gravity. This situation is analog to a forced harmonic oscillator

$$\ddot{x} + \frac{K}{m}x = F \,, \tag{4.8}$$

where the driving force F is due to gravity. The total force is $mF - Kx$, where x is the position of the oscillator and K is the force constant of the oscillator. The general solution for this system has two oscillatory modes with an angular frequency $w = \sqrt{K/m}$, and a particular solution is $x = F/w^2$. Assuming that the oscillator is initially at rest the sine mode vanishes, which leaves

$$x = A\cos(wt) + \frac{F}{w^2} \,. \tag{4.9}$$

The driving forces displaces the unforced situation from zero, so that the two extreme points at each side of the oscillations are not symmetric. The shift is more dramatic for smaller frequencies. The square of the oscillator position shows that the odd and even peaks have different heights due to this shift. Therefore, a forced harmonic oscillator is determined by the external force F and the reduced spring constant K/m.

In our case, for the photon-baryon fluid, the frequency grows as we decrease the effective mass of the fluid, i.e., as we decrease Ω_b. The fewer the baryon abundance, the higher the sound speed of the fluid (and closer the peaks of the wave pattern). In turn, with more cold dark matter, the gravitational potentials are larger, which increases the driving force (and lowers the frequency), and therefore the difference in the amplitude between odd and even peaks is larger. As the fluid falls in the potential, radiation pressure increases and pushes the plasma outwards to maximum expansion, leading to an underdensity with smaller amplitude than in the absence of gravity. Then the radiation pressure reduces and the plasma clusters again, and the cycle repeats from the beginning.

On a different note, remember that even if photon and baryons are tightly coupled, their interaction rate is not infinite. This allows the photons to travel a finite distance between two scatter events. The mean free path $\lambda_{\rm MFP}$ in this case is the inverse of the derivative of the optical depth, $\lambda_{\rm MFP} = (n_e\sigma_{\rm T}a)^{-1}$ from the collision term for photons. Over a Hubble time, photons undergo $\sim n_e\sigma_{\rm T}H^{-1}$ scatter events (transforming the scattering rate to time instead of conformal time, and multiplying for the time). For a random walk like this, the total distance traveled is the mean free path times the square root of the number of steps (i.e., scatter events). Therefore, a cosmological photon moves a mean comoving distance

$$\lambda_{\rm D} \sim \lambda_{\rm MFP}\sqrt{n_e\sigma_{\rm T}H^{-1}} = \left(a\sqrt{n_e\sigma_{\rm T}H}\right)^{-1} \tag{4.10}$$

over a Hubble time. Any perturbation on scales smaller than this distance will be washed out due to all the photons diffusing over a patch of this scale, which homogenizes the photon temperature. In Fourier space, this smoothing corresponds to a damping of high-k modes. Since $\lambda_{\rm D}$ depends on the number of electrons, the diffusion scale depends on Ω_b. Larger Ω_b reduces $\lambda_{\rm D}$ which in turn reduces the damping.

We have qualitatively described what is known as the primary CMB anisotropies. However, photons do not travel completely unaffected after recombination. Instead, they are affected by evolving gravitational potential (integrated Sachs-Wolfe effect), reionization, gravitational lensing due to metric perturbations along the line of sight, and interactions with free electrons (Sunyaev-Zeldovic effect).

We will provide a more accurate qualitative understanding of the photon perturbations to understand the CMB power spectrum and how we can use it to constrain cosmological parameters. As with the case of dark matter perturbations, an almost exact treatment requires the use of numerical Boltzmann code. On what follows, we use the Newtonian gauge, and as done in the previous chapter, we will distinguish between different regimes and stages of evolution to simplify the computations.

We will focus only on scalar perturbations and in the CMB temperature anisotropies. However, as discussed in the previous chapters, Compton scattering generates linear polarization (in turn, cosmological perturbations do not generate circular polarization). Nonetheless, only the quadrupole of the photon perturbations generate non-zero polarization. We can also distinguish between a curl-free, scalar component of the polarization (known as E mode) and a divergence-free, pseudoscalar component (known as B mode); scalar perturbations only generate E modes, while the B modes are generated either by primordial tensor perturbations or through secondary anisotropies like lensing. Since only the quadrupole generates polarization, $E(k) \propto \Theta_2(k)$ (in the tight-coupling approximation), and actually the monopole and quadrupole of the polarization perturbations contribute to the temperature perturbation through this same connection. Finally, note that the polarization perturbations must be significantly smaller than the temperature perturbations, since the quadrupole is suppressed in the early Universe due to Compton scattering.

4.1 Large-scale anisotropies

The large-scale limit can be treated with the same system that was discussed in Eq. (3.5). From the equation for the photon monopole, $\Theta_0' = -\Phi'$, we find that $\Theta_0 = -\Phi$ plus a constant. Similarly, from Eq. (2.64) we learn that the initial post-inflation condition is $\Theta_0 = \Phi/2$, so the constant must be $\mathcal{R} = 3\Phi_{\rm super\,hor.}/2$ (from Eq. (3.14)). The large-scale evolution of Φ is given by Eq, (3.13), but note that recombination takes places long after equality, hence $\Phi = 3\mathcal{R}/5$ in this limit. Therefore,

$$\Theta_0(\boldsymbol{k}, \tau_*) = -\Phi(\boldsymbol{k}, \tau_*) + \mathcal{R}(\boldsymbol{k}) = \frac{2}{5}\mathcal{R}(\boldsymbol{k}) = \frac{2}{3}\Phi(\boldsymbol{k}, \tau_*)\,. \qquad (4.11)$$

As discussed before, the observed anisotropy is $\Theta_0 + \Psi$ (and using that $\Psi \simeq -\Phi$), so that we have

$$(\Theta_0 + \Psi)(\boldsymbol{k}, \tau_*) = -\frac{1}{5}\mathcal{R}(\boldsymbol{k}) = -\frac{1}{3}\Phi(\boldsymbol{k}, \tau_*) \,. \tag{4.12}$$

From the last two equations we see something that may be counter intuitive. On the one hand, photons are hotter ($\Theta_0 > 0$) in places where gravity is more intense ($\Phi > 0$, $\Psi < 0$). However, we do not see them actually hotter, because the energy they lose as they climb those potential wells makes them actually cooler than those coming from places where gravity is less intense. This also applies for matter over and underdensities: if we integrate the equation for δ_c in Eq. (3.1) and apply the initial condition $\delta_c = \mathcal{R}$ from the inflation chapter, we find

$$\delta_c(\boldsymbol{k}, \tau_*) = \mathcal{R}(\boldsymbol{k}) - 3\left[\Phi(\boldsymbol{k}, \tau_*) - \frac{2}{3}\mathcal{R}(\boldsymbol{k})\right] = \frac{6}{5}\mathcal{R}(\boldsymbol{k}) = 2\Phi(\boldsymbol{k}, \tau_*) \,, \tag{4.13}$$

so that the observed anisotropy in terms of the dark matter overdensity is

$$(\Theta_0 + \Psi)(\boldsymbol{k}, \tau_*) = -\frac{1}{6}\delta_c(\boldsymbol{k}, \tau_*) \,, \tag{4.14}$$

presenting a similar behavior than with respecto to the gravitational potentials. Therefore, hotter observed anisotropies corresponds to underdense regions.

4.2 Baryon acoustic oscillations

The mean-free path of photons before recombination is significantly smaller than the size of the horizon, which couples them to baryons conforming a tightly-coupled photon-baryon fluid. This condition applies when the optical depth is $\gg 1$ (i.e., $\int n_e \sigma_{\rm T} a \gg 1$). As argued before, the competing forces of radiation pressure and gravity build acoustic oscillations in the fluid.

In this limit, all moments beyond the monopole and dipole are suppressed: the photons therefore behave like a fluid and can be described by its density and velocity. We can show this starting from Eqs. (1.137), and taking the limit in which $\lambda_{\rm MFP} = (n_e \sigma_{\rm T} a)^{-1}$ is very small. For the cases in which $\ell \geq 3$, $\Theta'_\ell \sim \Theta_\ell/\tau \ll n_e \sigma_{\rm T} a \Theta_\ell$, and neglecting the coupling to the higher multipole, we have

$$\Theta_\ell \sim \frac{k}{n_e \sigma_{\rm T} a}\frac{\ell}{2\ell+1}\Theta_{\ell-1} = k\lambda_{\rm MFP}\frac{\ell}{2\ell+1}\Theta_{\ell-1} \,. \tag{4.15}$$

Therefore, for scales much larger than the mean-free path, $\Theta_\ell \ll \Theta_{\ell-1}$ (which justifies neglecting of the higher multipole above). If we neglect the contribution from the difference between the two linear polarization components given by $\mathcal{G}_\ell$, we can also neglect Θ_2.

Physically, we can understand this as follows. Consider a plane-wave perturbation: an observer at its center sees photons coming from a distance $\sim \lambda_{\rm MFP}$. Therefore, large-scale perturbations (i.e., $k\lambda_{\rm MFP} \ll 1$) do not contribute to the perturbations that the observer perceives, because they produce a constant temperature over that volume. Small scales perturbations are in turn damped by the diffusion of the photons. Therefore, considering only the first two moments:

$$\begin{aligned}
&\Theta_0' + k\Theta_1 = -\Phi' \, , \\
&\Theta_1' - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} - a n_e \sigma_{\rm T} \left(\Theta_1 - \frac{\theta_b}{3k} \right) \, ,
\end{aligned} \tag{4.16}$$

which are accompanied by the baryon equations, which we can rewrite, defining $R \equiv 3\bar{\rho}_b / 4\bar{\rho}_\gamma$ and ignoring the acoustic term, as

$$\theta_b = 3k\Theta_1 - \frac{R}{n_e \sigma_{\rm T} a} \left(\mathcal{H}\theta_b - k^2\Psi + \theta_b' \right) \, . \tag{4.17}$$

The second term is much smaller due to the $R\lambda_{\rm MFP}$ factor (multiplied by $1/\tau$ and k in each case). To lowest order we take $\theta_b = 3k\Theta_1$, and expand substituting this lowest-order expression in the second term, leading to

$$\theta_b \simeq 3k\Theta_1 - \frac{R}{n_e \sigma_{\rm T} a} \left(3k\mathcal{H}\Theta_1 - k^2\Psi + 3k\Theta_1' \right) \, , \tag{4.18}$$

which we can use to eliminate θ_b in the photon perturbation equations above. After rearranging a bit the terms:

$$\Theta_1' + \frac{\mathcal{H}R}{1+R}\Theta_1 - \frac{k}{3(1+R)}\Theta_0 = \frac{k}{3}\Psi \, . \tag{4.19}$$

Now we have a system of two first-order equations; as done in the previous chapter we will differentiate the equation for Θ_0, substitute the equation above, and then use the equation for Θ_0 without differentiate to substitute Θ_1, to obtain

$$\Theta_0'' + \frac{\mathcal{H}R}{1+R}\Theta_0' + k^2 c_s^2 \Theta_0 = F(k,\tau) \, , \tag{4.20}$$

where we have defined the force function

$$F(k,\tau) \equiv -\frac{k^2}{3}\Psi - \frac{\mathcal{H}R}{1+R}\Phi' - \Phi'' \, , \tag{4.21}$$

and the sound speed of the fluid as

$$c_s(\tau) \equiv \sqrt{\frac{1}{3(1+R(\tau))}} \, . \tag{4.22}$$

Note that the sound speed depends on Ω_b. If the abundance of baryons is negligible, the sound speed tends to $1/\sqrt{3}$, as for any relativistic fluid. Baryons makes the fluid *heavier*, which acts as the inverse mass in the reduced spring constant of the forced harmonic oscillator. Actually, the equation above for Θ_0 is a forced, damped harmonic oscillator. Most of the terms multiplying Φ coincide with those of Θ_0 therefore we can rewrite the equation above as

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \frac{\mathcal{H}R}{1+R}\frac{\mathrm{d}}{\mathrm{d}\tau} + k^2 c_s^2\right\}\left[\Theta_0 + \Phi\right](\boldsymbol{k},\tau) = \frac{k^2}{3}\left[\frac{1}{1+R}\Phi - \Psi\right](\boldsymbol{k},\tau)\,. \tag{4.23}$$

We will use again the Green's method to solve the full solution, which proposes to find the particular solution starting from the two homogeneous general solutions. The drag term in the equation above goes as $R(\Theta_0 + \Phi)/\tau^2$ and the pressure ($\propto k^2 c_s^2$) is much larger for modes within the horizon or if R is small, which describes how for the scales of interest the impact of the pressure (oscillations, in this case) is much more significant than the one from the Hubble expansion. Although there is a solution including this term (the WKB solution, which assumes a solution of the $\Theta_0 = Ae^{iB}$), let us neglect the drag term, for which we have the oscillatory homogeneous solutions

$$S_1 = \sin(kr_s(\tau))\,; \qquad S_2 = \cos(kr_s(\tau))\,, \tag{4.24}$$

where the sound horizon is the comoving distance that the acoustic wave has had time to travel in time τ:

$$r_s = \int_0^\tau \mathrm{d}\tilde{\tau} c_s(\tilde{\tau})\,. \tag{4.25}$$

The total solution (including the particular solution for the driving force) can be obtained from these two solutions similarly than for Eq. (3.26) (and neglecting all instances of R outside the oscillatory homogeneous solutions):

$$\begin{aligned}\Theta_0 + \Phi =& C_1 S_1 + C_2 S_2 + \\ &+ \frac{k^2}{3}\int_0^\tau \mathrm{d}\tilde{\tau}(\Phi(\tilde{\tau}) - \Psi(\tilde{\tau}))\frac{S_1(\tilde{\tau})S_2(\tau) - S_1(\tau)S_2(\tilde{\tau})}{S_1(\tilde{\tau})S_2'(\tilde{\tau}) - S_1'(\tilde{\tau})S_2(\tilde{\tau})},.\end{aligned} \tag{4.26}$$

We can fix the integration constants to the initial condition for which both Θ_0 and Φ are constants. Therefore, the coefficient C_1 multiplying the sine must be zero, and $C_2(\boldsymbol{k}) = \Theta_0(\boldsymbol{k}, 0) + \Phi(\boldsymbol{k}, 0)$. In our limit, R is effectively very small, hence the denominator in the integral, which is $-kc_s$ can be approximated as $-k\sqrt{3}$. and the numerator can be reexpressed as the sine of the difference of the arguments, so that

$$\begin{aligned}\Theta_0 + \Phi =& (\Theta_0(0) + \Phi(0))\cos(kr_s) + \\ &+ \frac{k}{\sqrt{3}}\int_0^\tau \mathrm{d}\tilde{\tau}(\Phi(\tilde{\tau}) - \Psi(\tilde{\tau}))\sin\left[k(r_s(\tau) - r_s(\tilde{\tau})\right]\,.\end{aligned} \tag{4.27}$$

Since outside the horizon $\Theta_0 + \Phi$ is constant, only the cosine mode is excited and a clear oscillatory pattern can be appreciated in the solution. This expression can predict with accuracy the position of the acoustic peaks from a numerical solution. To get the solution we should numerically integrate the last term above, but we can simplify a bit a further. If the first term dominates, the position of the peaks is given by the extrema of $\cos(kr_s)$: $k_{\rm pk} = n\pi/r_s$, where n is a natural number, which is within 10% of the numerical solution.

Finally we can use Eq. (4.16) to relate this solution to the dipole of the photon distribution:

$$\begin{aligned}\Theta_1(\boldsymbol{k},\tau) = \frac{1}{\sqrt{3}}(\Theta_0(0)+\Phi(0))\sin(kr_s)- \\ -\frac{k}{3}\int_0^\tau \mathrm{d}\tilde{\tau}(\Phi(\tilde{\tau})-\Psi(\tilde{\tau}))\cos\left[k(r_s(\tau)-r_s(\tilde{\tau})\right]\,,\end{aligned} \tag{4.28}$$

which is completely out of phase with respect to the monopole, even after accounting for the integral term.

4.3 Diffusion damping

Diffusion is characterized by a small but non-negligible quadrupole moment. Therefore, we need to recover Eq. (1.137) to account for it to obtain the equivalent of Eq. (4.16). However, we can simplify on other end: diffusion matters at very small scales, where gravitational potentials are smaller than radiation perturbations by a factor $\mathcal{H}/k^2$. Otherwise, all the considerations made in the previous section still apply, so that we can neglect all moments above the quadrupole and we have (after neglecting the effects of polarization)

$$\begin{aligned}&\Theta_0' + k\Theta_1 = 0\,,\\ &\Theta_1' + \frac{k}{3}\left(2\Theta_2 - \Theta_0\right) = n_e\sigma_{\rm T}a\left(\frac{\theta_b}{3k} - \Theta_1\right)\,,\\ &\Theta_2' - \frac{2k}{5}\Theta_1 = -n_e\sigma_{\rm T}a\frac{9}{10}\Theta_2\,,\end{aligned} \tag{4.29}$$

along with

$$3k\Theta_1 - \theta_b = \frac{R}{n_e\sigma_{\rm T}a}\left(\mathcal{H}\theta_b + \theta_b'\right)\,, \tag{4.30}$$

which is a small rephrase of Eq. (4.17) after dropping the potentials. We know that the time dependence of the variables involved is gonna follow sinusoidal functions, hence let us assume that already, but using the exponential form, such as $\theta \propto e^{i\int \mathrm{d}\tilde{\tau}\omega}$, where we know that $\omega \simeq kc_s$ in the tight-coupled limit. This implies that the derivative with respect to conformal time is

$$|\theta_b'| = |i\omega\theta_b| \gg \mathcal{H}|\theta_b|\,, \tag{4.31}$$

where we have used the approximate value of ω and that $k \gg \mathcal{H}$ at small scales. Thus, we can drop the $\mathcal{H}\theta_b$ term in the baryon equation above. Substituting the relation between θ_b and θ_b' in the equation above and expanding the denominator up to second order, we have

$$\theta_b = 3k\Theta_1 \left[1 - \frac{i\omega R}{n_e\sigma_{\rm T}a} - \left(\frac{i\omega R}{n_e\sigma_{\rm T}a}\right)^2\right] . \tag{4.32}$$

We can do the same procedure for the quadrupole. First, since we are in a regime where the mean-free path is very small, hence we can drop the Θ_2' term, which leaves

$$\Theta_2 = \frac{4k}{9n_e\sigma_{\rm T}a}\Theta_1 , \tag{4.33}$$

which shows that our hierarchy closing scheme is sound: higher moments are suppressed by a $k\lambda_{\rm MFP}$ factor. Finally, the equation for the monopole is given by

$$i\omega\Theta_0 = -k\Theta_1 . \tag{4.34}$$

We can now insert all these expressions in the equation for the dipole, which returns the dispersion relation for ω (after collecting all the terms):

$$\omega^2(1+R) - \frac{k^2}{3} - \frac{i\omega}{n_e\sigma_{\rm T}a}\left[\omega^2R^2 + \frac{8k^2}{27}\right] = 0 . \tag{4.35}$$

Note that the last term is suppressed by a mean-free path factor. If we were to neglect that term, we would recover the result of the previous section: that the frequency is kc_s.[3] Since the last term is a correction, we can write the frequency of the oscillator as the previous result plus a minor correction

$$\delta\omega = \frac{ik^2}{2(1+R)n_e\sigma_{\rm T}a}\left[c_s^2R^2 + \frac{8}{27}\right] . \tag{4.36}$$

Therefore, the time dependence for the perturbations is given by

$$\sim \exp\left\{ik\int {\rm d}\tilde\tau c_s(\tilde\tau)\right\}\exp\left\{-\frac{k^2}{k_{\rm D}^2}\right\} , \tag{4.37}$$

where we have defined the damping scale and its corresponding wavenumber as

$$k_{\rm D}^{-2} \equiv \int_0^\tau \frac{{\rm d}\tilde\tau}{6(1+R)n_e\sigma_{\rm T}a(\tilde\tau)}\left[\frac{R^2}{1+R} + \frac{8}{9}\right] . \tag{4.38}$$

For an order-of-magnitude qualitative understanding, the above expression implies

$$\lambda_{\rm D} \sim k_{\rm D}^{-1} \sim \sqrt{\tau\lambda_{\rm MFP}} , \tag{4.39}$$

which matches our previous expectations (remember that $\tau \simeq \mathcal{H}^{-1}$). The diffusion scale grows with $\sim a^{1/2}$ and $\Omega_b^{-1/2}$, and damps the power spectrum at multipoles $\ell \gtrsim k_{\rm D}\tau_0 \sim 10^3$. This effect is known as the Silk damping.

[3]Note that in this case we do not have any forcing term in the harmonic oscillator because we have neglected the contribution from the gravitational potential.

4.4 Projection to anisotropies on the sky

Until now we have derived the three-dimensional perturbations in the photon-baryon fluid at recombination, but actually we are only sensitive to the projected anisotropies on the sky, once photons arrived to us. Remember that the moments were defined in terms of the angle between the direction of the propagation of the photon and $\boldsymbol{k}$, and that the direction of propagation is set by the fact that they arrive to us through a given line of sight. Therefore, we need a solution for the photon moments today in terms of the monopole and dipole at recombination.

We can use Eqs. (1.91) and (1.134), and rearrange a bit the terms to get

$$\Theta' + (ik\mu - \uptau')\Theta = \hat{S} \,, \tag{4.40}$$

where we have defined the scattering optical depth integrated backwards from today[4]

$$\uptau \equiv \int_{\tau_0}^{\tau} \mathrm{d}\tilde{\tau} n_e \sigma_{\mathrm{T}} a(\tilde{\tau}) \,, \qquad \uptau' = -n_e \sigma_{\mathrm{T}} a \,, \tag{4.41}$$

and the source function

$$\hat{S} \equiv -\Phi' - ik\mu\Psi - \uptau' \left[\Theta_0 - \frac{i\theta_b}{k}\mathcal{P}_1(\mu) - \frac{1}{2}\Pi\mathcal{P}_2(\mu) \right] \,, \tag{4.42}$$

in turn using

$$\Pi \equiv \frac{1}{4}\left(\mathcal{F}_{\gamma 2} + \mathcal{G}_{\gamma 0} + \mathcal{G}_{\gamma 2}\right) \,. \tag{4.43}$$

As a side note, it is now convention to set the *moment* of recombination τ_* as the conformal time for which $\uptau = 1$, although there are also alternative conventions. We can turn the differential equation above into an integral equation. Rewrite the left-hand side of Eq. (4.40) as a factor multiplying a time derivative so that

$$\Theta' + (ik\mu - \uptau')\Theta = \Theta' + \mathcal{A}\Theta = e^{-\mathcal{A}} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\Theta e^{\mathcal{A}}\right] \,. \tag{4.44}$$

Therefore, we can write $(\Theta e^{ik\mu\tau - \uptau})' = e^{ik\mu\tau-\uptau}\hat{S}$ and integrate over conformal time to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\Theta e^{\mathcal{A}}\right] = e^{\mathcal{A}}\hat{S} \implies \\ &\Theta(\tau_0) = \Theta(\tau_{\mathrm{init}}) e^{ik\mu(\tau_{\mathrm{init}} - \tau_0)} e^{-\uptau(\tau_{\mathrm{init}})} + \int_{\tau_{\mathrm{init}}}^{\tau_0} \mathrm{d}\tau \hat{S}(\tau) e^{ik\mu(\tau - \tau_0)} e^{-\uptau} \,, \end{aligned} \tag{4.45}$$

[4]Here we face a slight conflict regarding the notation. The optical depth is usually denoted by a regular τ. Here we decide to used the variant $\uptau$ to avoid confusion with the conformal time. Other sources, especially those that do not use the synchronous gauge, solve this conflict denoting the conformal time with η. On the other hand, there are references using κ to denote the optical depth; we prefer not to use that convention to avoid confusion with the curvature.

where we have used that $\tau(\tau_0) = 0$ by definition. On the other hand, $\tau(\tau_{\rm init})$ blows up for early enough times, so that the exponential vanishes and we can drop the first term. Conceptually, this corresponds to the fact that Compton scattering erases effectively any initial anisotropy. For the same reason, we can move $\tau_{\rm init}$ to 0 without any impact. Therefore, the solution for anisotropies is given by

$$\Theta(k, \mu, \tau_0) = \int_0^{\tau_0} \mathrm{d}\tau \hat{S}(k, \mu, \tau) e^{ik\mu(\tau-\tau_0)} e^{-\tau} \,. \tag{4.46}$$

We need to deal now with the dependence in μ, which is inside the source function and in the exponential. In the case of the exponential is easy, because we can multiply each side of the equation by a Legendre polynomial and remember that

$$\begin{aligned} (-i)^{-\ell} A_\ell &\equiv \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu \mathcal{P}_\ell(\mu) A \,, \\ (-i)^{-\ell} j_\ell(x) &\equiv \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu \mathcal{P}_\ell(\mu) e^{ix\mu} \,, \end{aligned} \tag{4.47}$$

so that it seems we could express the multipoles of Θ as function of Bessel function integrals. Also, not that $j_\ell(x) = (-1)^\ell j_\ell(-x)$.

Dealing with the μ dependence in the source function seems more complicated. However, since it multiplies the exponential, we can repeat the trick from the previous subsection and substitute each appearance it has by a time derivative on the rest of the term:

$$\mu \to \frac{1}{ik} \frac{\mathrm{d}}{\mathrm{d}\tau} \,, \qquad (\text{within}\, \hat{S}) \,. \tag{4.48}$$

We can do this for the all terms in which μ appears and use integration by parts to get the desired equality. For instance, for the $-ik\mu\Psi$ term:

$$\begin{aligned} -ik \int_0^{\tau_0} \mathrm{d}\tau \mu \Psi e^{ik\mu(\tau-\tau_0)} e^{-\tau} &= - \int_0^{\tau_0} \mathrm{d}\tau \Psi e^{-\tau} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[e^{ik\mu(\tau-\tau_0)} \right] = \\ &= \int_0^{\tau_0} \mathrm{d}\tau e^{ik\mu(\tau-\tau_0)} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\Psi e^{-\tau} \right] \,, \end{aligned} \tag{4.49}$$

where the last line is the result of the integration by parts after the surface term vanishes: the $e^{-\tau(0)}$ nulls all the term $\tau = 0$, and the $\tau = \tau_0$ term does not depend on μ, hence only affects the monopole of the CMB and we cannot detect it with the anisotropies.

This procedure can be applied similarly to the other term depending on μ as well as the one depending on the Legendre quadrupole (which involves a second derivative ($\mathcal{P}_2(\mu) = (3\mu^2 - 1)/2$). Accounting for all this, the solution is

$$\Theta_\ell(k, \tau_0) = \int_0^{\tau_0} \mathrm{d}\tau S(k, \tau) j_\ell \left[k(\tau_0 - \tau) \right] \tag{4.50}$$

with a new source function defined as

$$\begin{aligned} S(k,\tau) \equiv e^{-\tau}\left[-\Phi' - \tau'\left(\Theta_0 + \frac{1}{4}\Pi\right)\right] + \\ + \frac{\mathrm{d}}{\mathrm{d}\tau}\left[e^{-\tau}\left(\Psi - \frac{\theta_b \tau'}{k^2}\right)\right] - \frac{3}{4k^2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left[e^{-\tau}\tau'\Pi\right] . \end{aligned} \tag{4.51}$$

We can see that there are many factors in the source function that depend on $\tau' e^{-\tau}$. Thus, let us define the visibility function as a probability density that a photon scattered for the last time at a conformal time τ, given by

$$g(\tau) \equiv -\tau'(\tau) e^{-\tau(\tau)} , \tag{4.52}$$

and as it is easy to understand, g decays quickly after recombination since the Universe becomes neutral (numerically, it is due to the prefactor τ', the scattering rate, which gets reduced significantly as n_e decreases dramatically). Before recombination, photons scatter many times, so the visibility function is also very small. Therefore, the visibility function is a very sharp function and determines the *width* of recombination. An alternative convention to define τ_* is the time at which g peaks. For the level of precision attempted in this analytic understanding, both moments are roughly the same.

Neglecting the contribution from polarization (which is very small), the source function becomes

$$\begin{aligned} S(k,\tau) \simeq g(\tau)\left[\Theta_0(k,\tau) + \Psi)(k,\tau)\right] + \frac{1}{k^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g(\tau)\theta_b(k,\tau)\right] + \\ + e^{-\tau}\left[\Psi'(k,\tau) - \Phi'(k,\tau)\right] . \end{aligned} \tag{4.53}$$

Now in order to get an approximate analytical result, we can integrate Θ_ℓ over time, integrating the θ_b term by parts (where as above the surface term vanishes since $g(\tau) = 0$ in both ends):

$$\begin{aligned} \Theta_\ell(k,\tau_0) \simeq \int_0^{\tau_0} \mathrm{d}\tau g(\tau)\left[\Theta_0(k,\tau) + \Psi)(k,\tau)\right] j_\ell\left[k(\tau_0 - \tau)\right] - \\ - \frac{1}{k^2}\int_0^{\tau_0} \mathrm{d}\tau g(\tau)\theta_b(k,\tau) j_\ell'\left[k(\tau_0 - \tau)\right] + \\ + \int_0^{\tau_0} \mathrm{d}\tau e^{-\tau}\left[\Psi'(k,\tau) - \Phi'(k,\tau)\right] j_\ell\left[k(\tau_0 - \tau)\right] . \end{aligned} \tag{4.54}$$

The first two integrals are weighted by the visibility functions and are the dominant terms; the latter integral is weighted by $e^{-\tau}$ and only contributes for $\tau \lesssim 1$, which is true after recombination. Furthermore, the gravitational potentials are constant during matter domination, as we saw in the previous chapter, hence the last line will only contribute just after recombination, where radiation still has a small influence in the evolution of the potentials, and after dark energy becomes relevant. The last line is known as the integrated Sachs-Wolf effect, and the two contributions depending on the time are known as the early ISW and the late ISW, respectively.

The fact that the visibility function is so peaked simplifies significantly the first two integrals, the rest of the integrand of which varies at much lower rate. Therefore, we can evaluate them at τ_* and remove them from the integral, which is left to be only the integral of g which is 1 by definition. For instance $\int \mathrm{d}\tau g A = A_* \int \mathrm{d}\tau g = A_*$. Using the recursion relation to express j_ℓ' as function of $j_{\ell-1}$ and j_ℓ and that at τ_* we have $\theta_b = -3\Theta_1$ (from the discussion in previous sections), we obtain

$$\begin{aligned}\Theta_\ell(k,\tau_0) \simeq & \left[\Theta_0(k,\tau_*) + \Psi)(k,\tau_*)\right] j_\ell\left[k(\tau_0-\tau_*)\right] + \\ & + 3\Theta_1(k,\tau_*)\left(j_{\ell-1}\left[k(\tau_0-\tau_*)\right] - (\ell+1)\frac{j_\ell\left[k(\tau_0-\tau_*)\right]}{k(\tau_0-\tau_*)}\right) + \\ & + \int_0^{\tau_0} \mathrm{d}\tau e^{-\tau}\left[\Psi'(k,\tau) - \Phi'(k,\tau)\right] j_\ell\left[k(\tau_0-\tau)\right] .\end{aligned} \tag{4.55}$$

Each term is usually referred to as the monopole term, the dipole or Doppler term, and the ISW, respectively.

The expression above describes the scales where diffusion is not relevant. At smaller scales, since the diffusion scale changes very quickly around recombination, diffusion cannot be included just multiplying the $\Theta_0 + \Psi$ above by the damping. In turn, including the damping in the integral of the visibility function turns out to be a much better approximation. This adds a multiplicative factor in the first line of the expression above of

$$\int \mathrm{d}\tau g(\tau) e^{-k^2/k_\mathrm{D}^2(\tau)} . \tag{4.56}$$

These expressions agree with numerical solutions within 10% precision. We can see that these result matches the preliminary expectations at the beginning of the chapter. The monopole depends on $\Theta_0 + \Psi$, and the Bessel functions determine how much anisotropy on a given angular scale $\sim \ell^{-1}$ is contributed by a plane wave with wave number k. On very small angular scales where we can assume plane-parallel flat sky,

$$j_\ell(x) \rightarrow^{x/\ell \rightarrow 0} \frac{1}{\ell}\left(\frac{x}{\ell}\right)^{\ell-1/2} , \tag{4.57}$$

i.e., j_ℓ is extremely small for large ℓ if $x < \ell$, or, in our case, Θ_ℓ is very close to zero if $\ell > k\tau_0$. In essence, perturbations on scales k contribute predominantly to angular scales of order $\ell \sim k\tau_0$.

4.5 CMB angular power spectrum

Θ is the perturbation of the CMB characteristic temperature, but we can only observe it today and here (note that the small variation in the position due to the location of the satellites and the time period over all observations have been made are completely negligible). Time-ordered observations are collected in a map as function of position on the sky (an angle), rather than the three-dimensional direction of the incoming photon. This is just a mere change of variables and we can use either frame indistinguishably for denoting the position on the sky.

Therefore, we can expand the temperature perturbation in spherical harmonics as discussed at the beginning of the chapter, where the harmonic indices ℓ and m are the conjugate to the angular position. Also, note that in the flat sky approximation (valid for small angular scales), the harmonic transform can be understood as a 2D Fourier transform (by turning ℓ and m into a 2D vector $\boldsymbol{\ell}$). Thus, the maximum multipole that can be measured is related with the angular resolution of a given experiment. The total number of independent bits of information is given by the number $N_{\rm pix}$ of pixels in the map, which is also equivalent to the number of independent $a_{\ell m}$ coefficients. Therefore, since each multipole ℓ involves $2\ell+1$ m values, we can estimate the maximum multipole accessible by equating $\sum^{\ell_{\rm max}}(2\ell+1) = (\ell_{\rm max}+1)^2 = N_{\rm pix}$. Note that for the CMB there is another limitation to obtain information from very high ℓ besides the angular resolution of the experiment: at some point, the diffusion damping kills any correlation at very small scales and any measured correlation is due to foregrounds and secondary anisotropies.

By definition, the mean of a given coefficient $a_{\ell m}$ vanishes, and therefore we work with their covariance, their power spectrum. Recovering Eq. (4.6):

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'} \,. \tag{4.58}$$

All the measured coefficients are in practice samples for the same underlying distribution. Moreover, for each ℓ value there are $2\ell+1$ m components, so that higher ℓ values have more statistical precision regarding the determination of their underlying distribution. The uncertainty related with the fact that we can only measure one sky and cannot access more information than the $2\ell+1$ components is called the cosmic variance, which for the angular power spectrum scales as

$$\left(\frac{\sigma(C_\ell)}{C_\ell}\right)_{\rm cosmic\,variance} = \sqrt{\frac{2}{2\ell+1}} \,, \tag{4.59}$$

although partial scale coverage adds a factor of $f_{\rm sky}^{-1/2}$ to this estimation. Furthermore, the contamination from foregrounds (which is more difficult to control at larger scales) makes very complicated to reach the cosmic variance limit at the largest scales.

From the relation between $a_{\ell m}$ and Θ_ℓ from Eq. (4.5) we can compute the power spectrum. To compute the variance of the spherical harmonic coefficients we need to compute first the variance of $\Theta(\boldsymbol{k}, \tau_0)$, where we will drop the τ_0 dependence for simplicity. There are two different sources of correlation here: the primordial perturbations (random variable) and their evolution (deterministic process). This allows us (at linear level) to separate them using the transfer function as we did in the previous chapter. In this case we define the transfer

$$\mathcal{T}(\boldsymbol{k}, \hat{\boldsymbol{q}}) \equiv \frac{\Theta(\boldsymbol{k}, \hat{\boldsymbol{q}})}{\mathcal{R}(\boldsymbol{k})}\,, \tag{4.60}$$

which by definition is deterministic and can be removed from the ensemble average. Therefore,

$$\begin{aligned}\left\langle \Theta(\boldsymbol{k}, \hat{\boldsymbol{q}})\Theta^*(\boldsymbol{k}', \hat{\boldsymbol{q}}')\right\rangle &= \langle \mathcal{R}(\boldsymbol{k})\mathcal{R}^*(\boldsymbol{k}')\rangle\, \mathcal{T}(\boldsymbol{k}, \hat{\boldsymbol{q}})\mathcal{T}^*((\boldsymbol{k}', \hat{\boldsymbol{q}}') = \\ &= (2\pi)^3 \delta_D^{(3)}(\boldsymbol{k} - \boldsymbol{k}') P_{\mathcal{R}}(k)\mathcal{T}(\boldsymbol{k}, \hat{\boldsymbol{q}})\mathcal{T}^*((\boldsymbol{k}', \hat{\boldsymbol{q}}')\,.\end{aligned} \tag{4.61}$$

We have seen that for scalar perturbations what matters, rather than $(\boldsymbol{k}, \hat{\boldsymbol{q}})$ is (k, μ), so that we find that the power spectrum is (after integrating over $\boldsymbol{k}'$)

$$C_\ell = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P_{\mathcal{R}}(k) \int \mathrm{d}\Omega_\mathrm{q} Y^*_{\ell m}(\hat{\boldsymbol{q}})\mathcal{T}(k, \mu) \int \mathrm{d}\Omega'_\mathrm{q} Y_{\ell m}(\hat{\boldsymbol{q}}')\mathcal{T}^*(k, \mu')\,. \tag{4.62}$$

We can expand the transfer function as function of the Legendre polynomials as in Eq. (1.106) so that $\mathcal{T}_\ell = \Theta_\ell/\mathcal{R}$, which leaves

$$C_\ell = \frac{2}{\pi} \int \mathrm{d}k k^2 P_{\mathcal{R}}(k) |\mathcal{T}_\ell|^2\,, \tag{4.63}$$

where we have used the orthogonality of the Legendre polynomial and the normality of the spherical harmonics. For a given multipole, the power spectrum is an integral over all Fourier modes of the variance of Θ, and quantifies the variance of the distribution from which the $a_{\ell m}$ coefficients are drawn from. Let us walk over the different scale ranges in the CMB power spectrum.

Ultra-large-scale anisotropies trace perturbations that have entered our horizon only recently, providing a window to the initial conditions. In this regime we can neglect the dipole term in Θ_ℓ, which leaves $\Theta_0 + \Psi$ and the ISW. The former is known as the Sachs-Wolfe effect, and using Eq. (4.12) we have

$$C_\ell^{\mathrm{SW}} \simeq \frac{2}{25\pi} \int \mathrm{d}k k^2 P_{\mathcal{R}}(k) |j_\ell\left[k(\tau_0 - \tau_*)\right]|^2\,. \tag{4.64}$$

Substituting the expression of the primordial curvature power spectrum, neglecting τ_* in favor of τ_0 in the Bessel function, and changing the variable to $k\tau_0$, there is an analytic solution to the integral in terms of Gamma functions. If we further assume $n_s = 1$, they simplify and we find that

$$\ell(\ell+1) C_\ell^{\mathrm{SW}} \simeq \frac{8}{25} \mathcal{A}_s \tag{4.65}$$

is a constant, inherited from $k^3 P_{\mathcal{R}}$ being a constant if $n_s = 1$.[5] Deviations from this constant are due to the dipole term becoming relevant at higher ℓ and the late ISW effect –relevant at $\ell \lesssim 30$, since dark energy becomes relevant at $z \lesssim 1$ –(and, to a smaller degree, the red-tilt in the primordial curvature power spectrum). Nonetheless, the amplitude of the power spectrum at these scales can roughly give an idea of the value of $\mathcal{A}_s$.

As ℓ grows the power spectrum probes scales that are within the horizon at recombination, where the acoustic oscillations form and all the terms of Θ_ℓ matter. However, note that since a given value of ℓ has support from a given k range (selected by the Bessel function), we have now a series of peaks and troughs rather than peaks and zeros in the oscillatory pattern of the power spectrum. This also produces that the peak position is slightly shifted towards lower ℓ values, roughly $\ell_{\rm pk} \simeq 0.75\pi\tau_0/r_s$. The dipole term (which, as discussed before, is smaller than the monopole and out of phase with respect to it) contributes to raise all the power spectrum amplitude, but especially the one of the troughs. Notably, the monopole and dipole terms are uncorrelated (mathematically, this is due to the properties of the Bessel functions). Finally, there is a contribution from the early ISW: if we consider that the potentials evolve at time τ_c, all sub-horizon scales $k\tau_c > 1$ will be affected, which through the Bessel function translate to scales $\ell > (\tau_0 - \tau_c)/\tau_c$. Importantly, the early ISW is coherent with the monopole of the source term (i.e., they are proportional to the same Bessel function), which magnifies its impact in the power spectrum through their cross correlation.

So far we have assumed that photons completely free stream to us from the last-scattering surface. However, after reionization, electrons are free again and photons can scatter with them. Consider an optical depth $\tau_{\rm reio} \equiv \tau(\tau_{\rm late})$ to a time after recombination. As photons travel through those free electrons, only a fraction $e^{-\tau_{\rm reio}}$ escape and reach us, while a fraction $1 - e^{-\tau_{\rm reio}}$ scatters into the beam from all directions (thus any anisotropy that they had cancels out). This involves that for photons coming with a temperature $T(1+\Theta)$, we will measure

$$T(1+\Theta)e^{-\tau_{\rm reio}} + T(1-e^{-\tau_{\rm reio}}) = T(1+\Theta e^{-\tau_{\rm reio}})\,. \tag{4.66}$$

This effects only to scales within the horizon at reionization; only those with $\ell \gtrsim \tau_0/\tau_{\rm reio} \sim 100$ are affected. Reionization has a significantly larger impact in the polarization power spectrum.

[5] In the same way that $k^3 P(k)$ is the dimensionless power spectrum per logarithmic k bin, $\ell(\ell+1)C_\ell$ is the angular power spectrum per logarithmic interval in ℓ, and it is the common way to visually represent the angular power spectrum; in particular, we usually plot $\ell(2\ell+1)C_\ell/2\pi$.

4.6 Constraints on cosmological parameters

CMB anisotropies are the primary source of information to constrain cosmological parameters, given the great quality of the observations, how developed the theory is, the actual sensitivity to the cosmological parameters and the *simplicity* of the physics involved (e.g., known atomic physics, linear perturbations, Gaussianity, etc). In this section we will discuss the signatures of changing the values of the cosmological parameters in the CMB power spectrum. The main limitation of the CMB observations is the assumption of a cosmological model to translate recombination quantities into present-day parameters. Therefore, there are strong degeneracies with late-time parameters, such as those controlling the behavior of dark energy. This is why information from late-time probes such as those from the large-scale structure or distance measurements from supernovae are key to complement CMB observations, especially for beyond-ΛCDM models. In addition, considering the polarization anisotropies (their autocorrelation and their correlation with the temperature anisotropies) further increases the constraining power of the CMB.

We consider a curved ΛCDM model, including the curvature parameter Ω_k, Ω_Λ, $\Omega_c h^2$, $\Omega_b h^2$, n_s, $\mathcal{A}_s$, and the optical depth to reionization $\tau_{\rm reio}$. We consider the physical densities (instead of the density parameters themselves) since the CMB is significantly more sensitive to such parameter combinations than to them separately. The radiation density parameter is extremely well constrained by the CMB temperature measurement from FIRAS and is usually considered fixed in all analyses.

If the spatial section of the Universe is not flat geodesics do not follow straight lines and two photons parallel to each other will converge or diverge. The first impact is that the relations of angles from different lines of sight and the directions of the photons at the last scattering surface changes: a fixed physical scale (say, the first acoustic peak) projects onto a much smaller (larger) angular scale in an open (closed) Universe. This shifts the spectrum to higher (lower) ℓ values. This can be quantified in terms of the angular diameter distance to the last scattering surface. $\Omega_k \neq 0$ also increases the late ISW effect, but the impact is negligible to the allowed values of Ω_k. Current constraints on Ω_k imply deviations below 2×10^{-3}. Changing Ω_Λ (which since we keep $\Omega_m h^2$ fixed is similar to vary H_0 has a similar effect in the peak shift due to the change in the angular diameter distance, but with a much larger effect (not enough to be competitive) in the ISW.

Changing the amplitude $\mathcal{A}_s$ of primordial perturbations rescale the whole power spectrum in a scale-independent way, while changing n_s tilts all the power spectrum by the same slope, with small caveats due to the wide ℓ-range support of the Bessel functions at low ℓ. At the same time, the optical depth to reionization suppresses the amplitude of the power spectrum in such a way that for $\ell \gtrsim 100$ the amplitude of the power spectrum goes $\propto \mathcal{A}_s e^{-2\tau_{\rm reio}}$. Large-scale polarization anisotropies provide further information about $\tau_{\rm reio}$, and also the fact that the autocorrelation of polarization anisotropies does not have this suppression and that the temperature-polarization correlation amplitude goes as $\mathcal{A}_s e^{-\tau_{\rm reio}}$ helps to break the degeneracy. Multipoles at $\ell \lesssim 100$ are too affected by cosmic variance to help significantly to break this degeneracy. This is why the uncertainties in $\tau_{\rm reio}$ are the main limitation to increase the precision in $\mathcal{A}_s$ measurements.

Finally, changes in the cold dark matter and baryon abundances (compensating the change in matter abundance by changing Ω_Λ) induce a small relative shift in the peak location and change their amplitudes. Changing the baryon abundance change the sound horizon and therefore changes the oscillatory patterns. Furthermore, it changes the relative amplitude of odd and even peaks: their ratio of heights grows when there are more baryons. Baryons also change the diffusion scale by the increase of n_e: more baryons reduces the diffusion scale (increases $k_{\rm D}$). In turn, changing the cold dark matter abundance changes the driving term for the acoustic oscillations (i.e., because it depends on the gravitational potentials, determined by the cold dark matter perturbations), having a similar impact than changing the baryon density. It also dominates the determination of matter-radiation equality, which affects the growth of perturbations (more growth for more dark matter) and the early ISW (less ISW for more matter, since equality happens earlier and radiation is less relevant at recombination). This combination of effects allows for a very tight constraint on the baryon and cold dark matter abundances.

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